Decision Making Algorithm Based on Fermatean Fuzzy Schweizer-Sklar Power Average Operator and Its Application in Selection of Sustainable Health Care Waste Management Technique

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Abstract

Hospitals and other health care facilities produce health care waste (HCW), which poses risks to health care personnel, patients, the public, and the environment. Its complex makeup, including infectious pathogens and dangerous compounds, necessitates expert treatment to reduce health and environmental concerns. In numerous developing nations, healthcare waste disposal management has emerged as one of the most rapidly escalating concerns for urban towns and health care providers. Therefore, identifying the most sustainable HCW management technique (HCWMT) is a challenging endeavor due to the multitude of possibilities, criteria, and stringent governmental regulations

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governing HCW disposal. Therefore, this paper presents a multiattribute decision making (MADM) algorithm under the Fermatean fuzzy numbers (FFNs) environment to select the optimal HCWMT. To achieve this, we propose the Fermatean fuzzy Schweizer-Sklar power average (FFSSPA) aggregation operator (AO) and the Fermatean fuzzy Schweizer-Sklar power weighted average (FFSSPWA) AO for aggregating the FFNs by combining the features of power averaging AO and Schweizer-Sklar t-norm and t-conorm. The proposed FFSSPA AO and FFSSPA AO adjusts the influence of each input dynamically, taking into account its relative importance or reliability. However, based on the proposed FFSSPWA AO, we propose a MADM algorithm under the FFNs environment. Afterwards, we consider a mathematical case study for the assessment of sustainable HCWMTs and demonstrate the practical applicability of the proposed MADM algorithm. In this case study, five potential alternatives for sustainable health care waste management techniques (HCWMTs): “Mechanical Biological Treatment”, “Hydrothermal Carbonization”, “Incineration”, “Microwaving”, and “Chemical Disinfection”, which are evaluated based on seven attributes: “Environmental hazard”, “Health risk”, “Investment cost”, “Operation and maintenance cost”, “Revenue generation”, “Public acceptance”, and “Requirement of skilled labor”. The proposed algorithm identifies “Chemical Disinfection” as the most appropriate sustainable HCWMT for this case. Finally, we present two numerical examples to demonstrate the superiority and validity of the proposed MADM algorithm compared to existing MADM algorithms.

Keywords: Health care waste, Waste management, Fermatean fuzzy sets, Schweizer-Sklar norm, MADM.

1 Introduction

Every developing economy has faced significant challenges with regard to the management and treatment of waste from hospitals and other health care facilities. The environment and public health may be at risk from improper health care waste (HCW) management. According to Ho [16], if clinical waste is not properly handled, it might spread a number of fatal illnesses, such as HIV and hepatitis B and C. Approximately 85% of the waste produced by healthcare operations is regular, non-hazardous waste, just like domestic rubbish. The remaining 15% is categorized as hazardous substance, which could be carcinogenic, corrosive, reactive, infectious, radioactive, or flammable [8]. Despite the fact that some 16 billion vaccinations are administered globally each year, not all syringes and needles are properly disposed of after use. Thus, it can be said that improper medical waste management seriously jeopardizes public health. Because of the exponential development of HCW brought on by growing disease, managing it effectively has become extremely difficult, requiring creative solutions and sustainable methods to lessen the effects on the environment and human health. The most popular techniques for HCW management, such burning and dumping plastic waste, have detrimental consequences on groundwater, air quality, soil integrity and human health etc [9]. It is also critical to emphasize the necessity of sustainable activities like recycling and minimizing single-use items. Nonetheless, it is critical to recognize that recycling is not a complete solution to the problem. Investing in current treatment methods such as " Mechanical Biological Treatment", "Hydrothermal Carbonization", "Incineration", "Microwaving", and "Chemical Disinfection" is critical for effectively managing HCW and mitigating associated environmental and health risks. Healthcare specialists and municipalities face a strategic challenge in determining the best sustainable technique for HCW management. To choose the best sustainable HCW management technique (HCWMT) on the basis of the multiple criteria forming a typical multiattribute decision making (MADM) problem.

IN MADM process, to assess the HCWMTs, the main problem is to gathering correct data for HCWMTs under the different attributes due to restrictions, lack of knowledge, human error, and inconsistency in the situation. To address these challenges, fuzzy set (FS) theory [35] and its extensions, such as intuitionistic fuzzy set (IFS) [4], Pythagorean fuzzy set (PFS) [34] and Fermatean fuzzy sets (FFS) [30] have been widely used by the researchers. The FFS is the generalization of the IFS and PFS. The FFS provides a larger space compare to IFS and PFS to express the uncertainty due to its more flexible constraint ($0 \leq \alpha^3 + \beta^3 \leq 1$) of the membership degree α and non-membership degree β. Various decision making methods have been developed under these environment and we have explained them in the following.

Decision Making Methods Under the Fuzzy Sets and Fermatean Fuzzy Sets

Kumar and Garg [20] defined the TOPSIS method based on the set pair analysis theory under the intuitionistic fuzzy numbers (IFNs) environment. Hussain et al. [17] proposed the MADM method based on the proposed Aczel Alsina Heronian mean aggregation operator (AO) under the IFNs environment assessment of Solar Panel. Kumar and Chen [19] defined the improved aggregation operator based on Einstein norm and MADM method based on the proposed AO under the IFNs environment. Bhardwaj et al. [6] defined the MADM method based on the entropy measure in the context of IFNs. Amman et al. [3] proposed the MADM method based on the Spearman rank correlation coefficient under the FFNs environment. Akram et al. [1] presented the decision-making method for an effective sanitizer to reduce COVID-19 under FF environment. A MADM approach based on the Hamacher interactive geometric AO for FFS was developed by Shahzadi et al. [31]. Keshavarz-Ghorabaee et al. [18] developed the decision-making method based on WASPAS technique under the FFNs environment for the selection of green construction supplier selection. Aydemir and Yilmaz Gunduz [5] introduced the TOPSIS method with Dombi AOs for FFS and MADM problems based on them. Senapati and Yager [29] proposed the Fermatean fuzzy weighted average (FFWA) AO and Fermatean fuzzy power weighted average (FFPWA) AO and also developed the MADM algorithm under the FFNs environment. Garg et al. [14] proposed the Fermatean fuzzy Yager weighted average (FFYWA) AO and MADM method for the selection of authentic lab for the COVID-19 test. Chen et al. [10] developed the IWP-TOPSIS-GRA techniques for FFS and its application in healthcare waste treatment technology evalution. Alghazzawi et al. [2] defined the Fermatean fuzzy ordered weighted averaging (FFOWA) AO and MADM algorithm based on the proposed AO in the context of FFNs.

Decision Making Methods for Health Care Waste Management

Chauhan and Singh [8] provided a brief overviews related to the HCW management. Manupati et al. [22] defined the multicriteria decision making (MCDM) method for the evaluation of the HCWMTs. Chauhan and Singh [9] proposed the MADM method based on the analytic hierarchy process (AHP) and TOPSIS techniques under the fuzzy environment for the selection of the sustainable location for the HCW disposal. Mishra et al. [24] defined the EDAS approach based on the parametric divergence measure under the IFSs environment for the assessment of HCW disposal technology. Mishra and Rani [23] defined the MADM method based on the WASPAS technique under the FFNs context for the selection of sustainable location for the HCW disposal. Rao and Sujatha [26] proposed the WASPAS technique for the selection of best HCWMT. Debbarma et al. [11] proposed the SWARA–MABAC based decision making algorithm for the health care waste recycling. Pamučar et al. [25] proposed the decision making method under the fuzzy rough environment for the assessment of HCWMTs. Chakraborty and Saha [7] proposed the Bonferroni mean AO and decision making algorithm for the selection of HCWMTs under the FFNs environment. Gao et al. [12] defined the BMW-VIKOR based MADM method under the FFNs environment for the selection of best HCWMT.

1.1 Motivations of the Paper

This paper addresses the challenges associated with MADM problems and selecting the most efficient and sustainable methods in healthcare waste management under the FFNs environment. The primary motivations are:

(a) The optimal selection for sustainable HCWMT is categorized as a MADM problem, characterized by conflicting criteria.
(b) The existing FFWA AO [29], FFPWA AO [29], FFOWA AO [2] and FFYWA AO [14] are not flexible with the parameters and not also remove the influence of each input based on its relative importance or reliability.
(c) We find that MADM algorithm based on FFWA AO [29], MADM algorithm based on FFPWA AO [29], MADM algorithm based on FFOWA AO [2] and MADM algorithm based on FFYWA AO [14] cannot distinguish the ranking order (RO) of the alternatives in some cases.
(d) Therefore, to overcome the limitations of the existing MADM algorithms given in [2, 14, 29], we need to develop a new MADM algorithm under the FFNs environment.

1.2 Contributions and Novelty of the Paper

In this study, we propose the Fermatean fuzzy Schweizer-Sklar power average (FFSSPA) AO and the Fermatean fuzzy Schweizer-Sklar power weighted average (FFSSPWA) AO for aggregating the FFNs by using power averaging AO and Schweizer-Sklar t-norm and t-conorm. The parameter-adjustable Schweizer-Sklar norms can switch between disjunctive (OR-like) and conjunctive (AND-like) aggregation behaviors. The ability to adapt allows decision-makers to accurately represent a wide range of real-life situations. Depending on how important or reliable each input is, the power averaging AO dynamically adjusts its influence. Compared to fixed-weight aggregating operators, this method more accurately represents the real impact of each criterion. We also present proofs of their desirable characteristics to show the validity of the proposed AOs. Afterwards, we develop a MADM algorithm under the FFNs environment by using FFSSPWA AO. Furthermore, to illustrate the applicability of the proposed MADM algorithm, we consider a mathematical case study of the selection of optimal HCW management techniques (HCWMTs) for sustainable management and disposal of HCW. In this case study, the proposed MADM algorithm selects the optimal HCWMT among the five HCWMTs " Mechanical Biological Treatment" (A_1), "Hydrothermal Carbonization" (A_2), "Incineration" (A_3), "Microwaving" (A_4), and "Chemical Disinfection" (A_5) under the seven attributes "Environmental hazard" (G_1), "Health risk" (G_2), "Investment cost" (G_3), "Operation and maintenance cost" (G_4), "Revenue generation" (G_5), "Public acceptance" (G_6) and "Requirement of skilled labor" (G_7). We also provide a comparison of the obtained ranking orders (ROs) using the proposed MADM algorithm with ROs obtained by the MADM algorithm based on FFWA AO [29], MADM algorithm based on FFPWA AO [29], MADM algorithm based on FFOWA AO [2] and MADM algorithm based on FFYWA AO [14]. Finally, we consider two numerical examples to demonstrate the superiority and robustness of the proposed MADM algorithm compared to MADM algorithm based on FFWA AO [29], MADM algorithm based on FFPWA AO [29], MADM algorithm based on FFOWA AO [2] and MADM algorithm based on FFYWA AO [14]. The proposed MADM algorithm can address the shortcomings of the MADM algorithm based on FFWA AO [29], MADM algorithm based on FFPWA AO [29], MADM algorithm based on FFOWA AO [2] and MADM algorithm based on FFYWA AO [14], where they cannot distinguish the ROs of the alternatives in some scenarios.

1.3 Arrangement of the Paper

The rest of the paper is organized as follows: Section 2 presents the foundational concepts relevant to this study. Section 3 introduces the power aggregation operators using Schweizer-Sklar t-norm, t-conorm and power averaging AO, and proves their desirable properties. In Section 4, we propose a MADM algorithm based on the proposed FFSSPWA AO. Section 5 provides a case study of selecting sustainable HCWMTs. Section 6 provides the superiority of the proposed MADM algorithm over the existing MADM algorithms. Finally, Section 7 provides a comprehensive summary of the paper. Figure 1 shows a graphical abstract of this study.

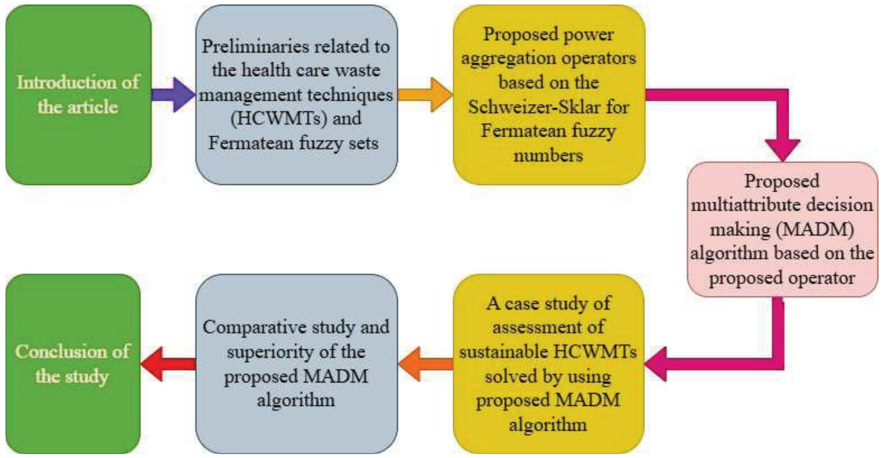

Figure 1 Graphical structure of the article outline.

2 Preliminaries

Definition 2.1. [30] A Fermatean fuzzy set $\tilde{F}$ in finite universe of discourse X is defined as follows:

$$\tilde{F} = \{\langle z, \alpha_{\tilde{F}}(z), \beta_{\tilde{F}}(z) \mid z \in X\rangle\}, \tag{1}$$

where $\alpha_{\tilde{F}}(z) : X \to [0,1]$ represents the membership degree of $z \in X$ and $\beta_{\tilde{F}}(z) : X \to [0,1]$ represents the non-membership degree of $z \in X$ such that $0 \le \alpha^3_{\tilde{F}}(z) + \beta^3_{\tilde{F}}(z) \le 1$ for all $z \in X$. The hesitancy degree of z in X is given by $\pi_{\tilde{F}} = \sqrt[3]{1 - \alpha^3_{\tilde{F}}(z) - \beta^3_{\tilde{F}}(z)}$. Usually, the pair $(\alpha_{\tilde{F}}(z), \beta_{\tilde{F}}(z))$ is called FFN.

Definition 2.2. [30] The score function s of a FFN $z = (\alpha, \beta)$ is defined as follows:

$$s(z) = \alpha^3 - \beta^3, \tag{2}$$

where $s(z) \in [-1, 1]$.

Definition 2.3. [30] The accuracy function Ψ of a FFN $z = (\alpha, \beta)$ is defined as follows:

$$\Psi(z) = \alpha^3 + \beta^3, \tag{3}$$

where $\Psi(z) \in [0, 1]$.

Definition 2.4. [30] If $z_1 = (\alpha_1, \beta_1)$ and $z_2 = (\alpha_2, \beta_2)$ be any two FFNs, then

(1) If $s(z_1) > s(z_2)$ then $z_1 \succ z_2$;
(2) If $s(z_1) < s(z_2)$ then $z_1 \prec z_2$;
(3) If $s(z_1) = s(z_2)$ then

 (i) If $\Psi(z_1) > \Psi(z_2)$ then $z_1 \succ z_2$;
 (ii) If $\Psi(z_1) < \Psi(z_2)$ then $z_1 \prec z_2$;
 (iii) If $\Psi(z_1) = \Psi(z_2)$, then $z_1 = z_2$.

Definition 2.5. [33] The power average (PA) AO for real numbers $\xi_1, \xi_2, \ldots, \xi_n$ is defined as follows:

$$PA(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{i=1}^{n} \frac{1 + T(\xi_i)}{\sum_{i=1}^{n}(1 + T(\xi_i))} \xi_i, \tag{4}$$

where $T(\xi_i) = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} Sup(\xi_i, \xi_j)$, and $Sup(\xi_i, \xi_j)$ denotes the support degree for ξ_i from ξ_j, which satisfies the following properties:

(i) $Sup(\xi_i, \xi_j) \in [0, 1]$;
(ii) $Sup(\xi_i, \xi_j) = Sup(\xi_j, \xi_i)$;
(iii) $Sup(\xi_i, \xi_j) \geq Sup(\xi_u, \xi_v)$ if $d(\xi_i - \xi_j) \leq d(\xi_u - \xi_v)$.

Definition 2.6. [28] Let x and y be two real numbers and $\eta < 0$. The Schweizer-Sklar's t-norm T and t-conorm U are defined as follows:

$$T(x, y) = (x^\eta + y^\eta - 1)^{\frac{1}{\eta}},$$
$$U(x, y) = 1 - ((1 - x)^\eta + (1 - y)^\eta - 1)^{\frac{1}{\eta}}.$$

Definition 2.7. [32] Let $z_1 = (\alpha_1, \beta_1)$, $z_2 = (\alpha_2, \beta_2)$ be any two FFNs. The operation laws of FFNs based on Schweizer-Sklar t-norm and t-conorm are defined as follows:

(i) $z_1 \bigoplus z_2 = (\sqrt[3]{1 - ((1 - \alpha_1^3)^\eta + (1 - \alpha_2^3)^\eta - 1)^{\frac{1}{\eta}}}, \sqrt[3]{(\beta_1^{3\eta} + \beta_2^{3\eta} - 1)^{\frac{1}{\eta}}})$;
(ii) $z_1 \bigotimes z_2 = (\sqrt[3]{(\alpha_1^{3\eta} + \alpha_2^{3\eta} - 1)^{\frac{1}{\eta}}}, \sqrt[3]{1 - ((1 - \beta_1^3)^\eta + (1 - \beta_2^3)^\eta - 1)^{\frac{1}{\eta}}})$;

(iii) $\lambda z_1 = (\sqrt[3]{1-(\lambda(1-\alpha_1^3)^\eta-(\lambda-1))^{\frac{1}{\eta}}}, \sqrt[3]{(\lambda\beta_1^{3\eta}-(\lambda-1))^{\frac{1}{\eta}}});$

(iv) $z_1^\lambda = (\sqrt[3]{(\lambda\alpha_1^{3\eta}-(\lambda-1))^{\frac{1}{\eta}}}, \sqrt[3]{1-(\lambda(1-\beta_1^3)^\eta-(\lambda-1))^{\frac{1}{\eta}}},);$

where $\lambda > 0$ and $\eta < 0$.

3 Proposed Schweizer-Sklar Power Aggregation Operators for FFNs

In this segment, we propose Fermatean fuzzy Schweizer-Sklar power averaging (FFSSPA) AO and Fermatean fuzzy Schweizer-Sklar power weighted averaging (FFSSPWA) AO based on the Schweizer-Sklar operations presented in Definition 2.7 and the PA AO shown in Definition 2.5.

Definition 3.1. Let $z_1, z_2, \ldots, z_n$ be a collection of FFNs. The proposed FFSSPA AO for aggregating the FFNs $z_1, z_2, \ldots, z_n$ is defined as follows:

$$FFSSPA(z_1, z_2, \ldots, z_n) = \bigoplus_{t=1}^{n} \frac{(1+T(z_t))}{\sum_{t=1}^{n}(1+T(z_t))} z_t, \tag{5}$$

where $T(z_t) = \sum_{\substack{t=1 \\ t\neq k}}^{n} S(z_t, z_k)$, $S(z_t, z_k) = 1 - \left(\frac{1}{2}\left\{ \left(\sqrt{\alpha_t^3}-\sqrt{\alpha_k^3}\right)^2 + \left(\sqrt{\beta_t^3}-\sqrt{\beta_k^3}\right)^2 + \left(\sqrt{\pi_t^3}-\sqrt{\pi_k^3}\right)^2 \right\}\right)^{1/2}$ represents the support for z_t from z_k, which satisfies the following properties:

(i) $S(z_t, z_k) \in [0, 1]$,
(ii) $S(z_t, z_k) = S(z_k, z_t)$,
(iii) $S(z_t, z_k) \geq S(z_p, z_q)$ if $|z_t - z_k| < |z_p - z_q|$.

Theorem 3.1. *For the FFNs $z_1 = (\alpha_1, \beta_1)$, $z_2 = (\alpha_2, \beta_2)$, …, $z_n = (\alpha_n, \beta_n)$, their aggregated value by using the proposed FFSSPA AO is a FFN and provided as follows:*

$$FFSSPA(z_1, z_2, \ldots, z_n) = \left(\sqrt[3]{1-(\sum_{t=1}^{n}\delta_t(1-\alpha_t^3)^\eta - \sum_{t=1}^{n}\delta_t+1)^{\frac{1}{\eta}}}, \sqrt[3]{\left(\sum_{t=1}^{n}\delta_t\beta_t^{3\eta} - \sum_{t=1}^{n}\delta_t+1\right)^{\frac{1}{\eta}}}\right), \tag{6}$$

where $\eta < 0$, $\delta_t = \frac{(1+T(z_t))}{\sum_{t=1}^{n}(1+T(z_t))}$, $T(z_t) = \sum_{\substack{t=1 \\ t\neq k}}^{n} S(z_t, z_k)$.

Proof 3.1. Let $z_1 = (\alpha_1, \beta_1)$, $z_2 = (\alpha_2, \beta_2)$, ..., $z_n = (\alpha_n, \beta_n)$ be FFNs, $\eta < 0$ and $\delta_t = \frac{(1+T(z_t))}{\sum_{t=1}^{n}(1+T(z_t))}$. To demonstrate this theorem, we use the mathematical induction principle as follows:

(i) Take $t = 2$, then by using Definition 2.7, we get

$$\delta_1 z_1 = \left(\sqrt[3]{1-\left(\delta_1\big(1-\alpha_1^3\big)^{\eta}-\delta_1+1\right)^{\frac{1}{\eta}}}, \sqrt[3]{\left(\delta_1\beta_1^{3\eta}-\delta_1+1\right)^{\frac{1}{\eta}}}\right)$$

$$\delta_2 z_2 = \left(\sqrt[3]{1-\left(\delta_2\big(1-\alpha_2^3\big)^{\eta}-\delta_2+1\right)^{\frac{1}{\eta}}}, \sqrt[3]{\left(\delta_2\beta_2^{3\eta}-\delta_2+1\right)^{\frac{1}{\eta}}}\right)$$

$$\begin{aligned}
&FFSSPA(z_1, z_2)\\
&= \delta_1 z_1 \bigoplus \delta_2 z_2\\
&= \left(\sqrt[3]{1-\left(\delta_1\big(1-\alpha_1^3\big)^{\eta}-\delta_1+1\right)^{\frac{1}{\eta}}}, \sqrt[3]{\left(\delta_1\beta_1^{3\eta}-\delta_1+1\right)^{\frac{1}{\eta}}}\right)\\
&\quad \bigoplus \left(\sqrt[3]{1-\left(\delta_2\big(1-\alpha_2^3\big)^{\eta}-\delta_2+1\right)^{\frac{1}{\eta}}}, \sqrt[3]{\left(\delta_2\beta_2^{3\eta}-\delta_2+1\right)^{\frac{1}{\eta}}}\right)\\
&= \left(\sqrt[3]{\begin{array}{l} 1-\Bigg(\left(1-1+\big(\delta_1\big(1-\alpha_1^3\big)^{\eta}-\delta_1+1\big)^{\frac{1}{\eta}}\right)^{\eta} \\ +\left(1-1+\big(\delta_2\big(1-\alpha_2^3\big)^{\eta}-\delta_2+1\big)^{\frac{1}{\eta}}\right)^{\eta}-1\Bigg)^{\frac{1}{\eta}} \end{array}},\right.\\
&\quad \left.\sqrt[3]{\left(\left(\big(\delta_1\beta_1^{3\eta}-\delta_1+1\big)^{\frac{1}{\eta}}\right)^{\eta}+\left(\big(\delta_2\beta_2^{3\eta}-\delta_2+1\big)^{\frac{1}{\eta}}\right)^{\eta}-1\right)^{\frac{1}{\eta}}}\right)\\
&= \left(\sqrt[3]{1-\left(\big(\delta_1\big(1-\alpha_1^3\big)^{\eta}-\delta_1+1\big)+\big(\delta_2\big(1-\alpha_2^3\big)^{\eta}-\delta_2+1\big)-1\right)^{\frac{1}{\eta}}},\right.
\end{aligned}$$

$$\sqrt[3]{\left(\left(\delta_1\beta_1^{3\eta}-\delta_1+1\right)+\left(\delta_2\beta_2^{3\eta}-\delta_2+1\right)-1\right)^{\frac{1}{\eta}}}\Bigg)$$

$$=\left(\sqrt[3]{1-\left(\delta_1\left(1-\alpha_1^3\right)^{\eta}+\delta_2\left(1-\alpha_2^3\right)^{\eta}-\delta_1-\delta_2+1\right)^{\frac{1}{\eta}}},\right.$$

$$\left.\sqrt[3]{\left(\delta_1\beta_1^{3\eta}+\delta_2\beta_2^{3\eta}-\delta_1-\delta_2+1\right)^{\frac{1}{\eta}}}\right)$$

$$=\left(\sqrt[3]{1-\left(\sum_{t=1}^{2}\delta_t\left(1-\alpha_t^3\right)^{\eta}-\sum_{t=1}^{2}\delta_t+1\right)^{\frac{1}{\eta}}},\right.$$

$$\left.\sqrt[3]{\left(\sum_{t=1}^{2}\delta_t\beta_t^{3\eta}-\sum_{t=1}^{2}\delta_t+1\right)^{\frac{1}{\eta}}}\right).$$

Thus, the result in Equation (6) is true for $t=2$.

(ii) Now, assume the result is true for $t=m$.

$$FFSSPA(z_1,z_2,\ldots,z_m)$$
$$=\delta_1 z_1\bigoplus\delta_2 z_2\bigoplus\ldots\bigoplus\delta_m z_m$$
$$=\left(\sqrt[3]{1-\left(\sum_{t=1}^{m}\delta_t\left(1-\alpha_t^3\right)^{\eta}-\sum_{t=1}^{m}\delta_t+1\right)^{\frac{1}{\eta}}},\right.$$
$$\left.\sqrt[3]{\left(\sum_{t=1}^{m}\delta_t\beta_t^{3\eta}-\sum_{t=1}^{m}\delta_t+1\right)^{\frac{1}{\eta}}}\right).$$

(iii) Now, consider $t=m+1$, we get

$$FFSSPA(z_1,z_2,\ldots,z_{m+1})$$
$$=\bigoplus_{t=1}^{m}\delta_t z_t\bigoplus\delta_{m+1}z_{m+1}$$

$$= \left(\sqrt[3]{1 - \left(\sum_{t=1}^{m} \delta_t \big(1 - \alpha_t^3\big)^\eta - \sum_{t=1}^{m} \delta_t + 1 \right)^{\frac{1}{\eta}}}, \right.$$

$$\left. \sqrt[3]{\left(\sum_{t=1}^{m} \delta_t \beta_t^{3\eta} - \sum_{t=1}^{m} \delta_t + 1 \right)^{\frac{1}{\eta}}} \right)$$

$$\bigoplus \left(\sqrt[3]{1 - \big(\delta_{m+1}(1 - \alpha_{m+1}^3)^\eta - (\delta_{m+1} - 1)\big)^{\frac{1}{\eta}}}, \right.$$

$$\left. \sqrt[3]{\Big(\delta_{m+1}\beta_{m+1}^{3\eta} - (\delta_{m+1} - 1)\Big)^{\frac{1}{\eta}}} \right)$$

$$= \left(\sqrt[3]{\begin{matrix} 1 - \Bigg(\Bigg(\Bigg(1 - 1 + \left(\sum_{t=1}^{m} \delta_t \big(1 - \alpha_t^3\big)^\eta - \sum_{t=1}^{m} \delta_t + 1 \right)^{\frac{1}{\eta}} \Bigg)^\eta \\ + \Big(1 - 1 + \big(\delta_{m+1}\big(1 - \alpha_{m+1}^3\big)^\eta - \delta_{m+1} + 1\big)^{\frac{1}{\eta}}\Big)^\eta - 1 \Bigg)^{\frac{1}{\eta}} \end{matrix}}, \right.$$

$$\left. \sqrt[3]{\begin{matrix} \Bigg(\Bigg(\left(\sum_{t=1}^{m} \delta_t \beta_t^{3\eta} - \sum_{t=1}^{m} \delta_t + 1 \right)^{\frac{1}{\eta}} \Bigg)^\eta \\ + \left(\left(\delta_{m+1}\beta_{m+1}^{3\eta} - \delta_{m+1} + 1 \right)^{\frac{1}{\eta}} \right)^\eta - 1 \Bigg)^{\frac{1}{\eta}} \end{matrix}} \right)$$

$$= \left(\sqrt[3]{1 - \left(\sum_{t=1}^{m+1} \delta_t (1 - \alpha_t^3)^\eta - \sum_{t=1}^{m+1} \delta_t + 1 \right)^{\frac{1}{\eta}}}, \right.$$

$$\left. \sqrt[3]{\left(\sum_{t=1}^{m+1} \delta_t \beta_t^{3\eta} - \sum_{t=1}^{m+1} \delta_t + 1 \right)^{\frac{1}{\eta}}} \right).$$

Thus, the result in Equation (6) holds for $t = m + 1$.

Hence, the result in Equation (6) is true for all natural numbers.

Example 3.1. Let $z_1 = (0.5, 0.4)$, $z_2 = (0.6, 0.1)$ and $z_3 = (0.4, 0.4)$ be any three FFNs. First, we calculate the supports values $S(z_t, z_k)$ between the FFNs z_t and z_k as follows:

$$S(z_1, z_2) = 1 - \left(\sqrt{\frac{1}{2} \begin{bmatrix} \left(\sqrt{(0.5)^3} - \sqrt{(0.6)^3}\right)^2 \\ + \left(\sqrt{(0.4)^3} - \sqrt{(0.1)^3}\right)^2 \\ + \left(\sqrt{(0.9326)^3} - \sqrt{(0.9217)^3}\right)^2 \end{bmatrix}} \right)$$
$$= 0.8245,$$
$$S(z_1, z_3) = 1 - \left(\sqrt{\frac{1}{2} \begin{bmatrix} \left(\sqrt{(0.5)^3} - \sqrt{(0.4)^3}\right)^2 \\ + \left(\sqrt{(0.4)^3} - \sqrt{(0.4)^3}\right)^2 \\ + \left(\sqrt{(0.9326)^3} - \sqrt{(0.9554)^3}\right)^2 \end{bmatrix}} \right)$$
$$= 0.9251,$$
$$S(z_2, z_3) = 1 - \left(\sqrt{\frac{1}{2} \begin{bmatrix} \left(\sqrt{(0.6)^3} - \sqrt{(0.4)^3}\right)^2 \\ + \left(\sqrt{(0.1)^3} - \sqrt{(0.4)^3}\right)^2 \\ + \left(\sqrt{(0.9217)^3} - \sqrt{(0.9554)^3}\right)^2 \end{bmatrix}} \right)$$
$$= 0.7806.$$

After that, we calculate $T(z_1), T(z_2)$ and $T(z_3)$ of the FFNs z_1, z_2 and z_3, respectively, where $T(z_t) = \sum_{\substack{t=1 \\ t \neq k}}^{n} S(z_t, z_k)$,

$$T(z_1) = S(z_1, z_2) + S(z_1, z_3) = 0.8245 + 0.9251 = 1.7496,$$
$$T(z_2) = S(z_2, z_1) + S(z_2, z_3) = 0.8245 + 0.7806 = 1.6051,$$

$$T(z_3) = S(z_3, z_1) + S(z_3, z_2) = 0.9251 + 0.7806 = 1.7057.$$

Now, we calculate δ_1, δ_2 and δ_3 for the FFNs z_1, z_2 and z_3, respectively, where $\delta_1 = \frac{(1+T(z_1))}{\sum_{t=1}^{3}(1+T(z_t))} = 0.3411$, $\delta_2 = 0.3232$ and $\delta_3 = 0.3357$.

Now, by using Equation (6), we aggregate the FFNs z_1, z_2 and z_3, for $\eta = -3$,

$$\begin{aligned} \text{FFSSPA}(z_1, z_2, z_3) &= \left(\sqrt[3]{\begin{array}{l} 1 - \Big(0.3411\left(1-(0.5)^3\right)^{-3} \\ \quad + 0.3232\left(1-(0.6)^3\right)^{-3} \\ \quad + 0.3357\left(1-(0.4)^3\right)^{-3} - 1 + 1\Big)^{\frac{1}{-3}} \end{array}}, \right. \\ &\qquad \left. \sqrt[3]{\begin{array}{l} \Big(0.3411\,(0.4)^{3(-3)} + 0.3232\,(0.1)^{3(-3)} \\ \quad + 0.3357\,(0.4)^{3(-3)} - 1 + 1\Big)^{\frac{1}{-3}} \end{array}} \right), \\ &= (0.5230, 0.1134). \end{aligned}$$

Property 3.1 (Idempotency). Let $z_1 = (\alpha_1, \beta_1)$, $z_2 = (\alpha_2, \beta_2)$, ..., $z_n = (\alpha_n, \beta_n)$ be FFNs. If $z_1 = z_2 = \ldots = z_n = z = (\alpha, \beta)$, then $\text{FFSSPA}(z_1, z_2, \ldots, z_n) = z$.

Proof 3.2.

$$\begin{aligned} FFSSPA(z_1, z_2, \ldots, z_n) &= \left(\sqrt[3]{1 - \left(\sum_{t=1}^{n} \delta_t \big(1-\alpha_t^3\big)^{\eta} - \sum_{t=1}^{n} \delta_t + 1 \right)^{\frac{1}{\eta}}}, \right. \\ &\qquad \left. \sqrt[3]{\left(\sum_{t=1}^{n} \delta_t \beta_t^{3\eta} - \sum_{t=1}^{n} \delta_t + 1 \right)^{\frac{1}{\eta}}} \right) \\ &= \left(\sqrt[3]{1 - \left(\sum_{t=1}^{n} \delta_t \big(1-\alpha^3\big)^{\eta} - \sum_{t=1}^{n} \delta_t + 1 \right)^{\frac{1}{\eta}}}, \right. \end{aligned}$$

$$\sqrt[3]{\left(\sum_{t=1}^{n}\delta_t\beta^{3\eta}-\sum_{t=1}^{n}\delta_t+1\right)^{\frac{1}{\eta}}}\Bigg)$$

$$=\left(\sqrt[3]{1-((1-\alpha^3)^{\eta}-1+1)^{\frac{1}{\eta}}},\right.$$

$$\left.\sqrt[3]{(\beta^{3\eta}-1+1)^{\frac{1}{\eta}}}\right)$$

$$=(\alpha,\beta)=z.$$

Property 3.2 (Monotonicity). Let $z_t=(\alpha_t,\beta_t)$ and $\hat{z_t}=(\hat{\alpha_t},\hat{\beta_t})$, $t=1,2,\ldots,n$, be two collections of FFNs. If $z_t\le\hat{z_t}$, $\forall t\ (t=1,2,\ldots,n)$ then

$$FFSSPA(z_1,z_2,\ldots,z_n)\le FFSSPA(\hat{z_1},\hat{z_2},\ldots,\hat{z_n}).$$

Proof 3.3. By using Equation (6), we get

$$FFSSPA(z_1,z_2,\ldots,z_n)$$
$$=\left(\sqrt[3]{1-\left(\sum_{t=1}^{n}\delta_t(1-\alpha_t^3)^{\eta}-\sum_{t=1}^{n}\delta_t+1\right)^{\frac{1}{\eta}}},\right.$$
$$\left.\sqrt[3]{\left(\sum_{t=1}^{n}\delta_t\beta_t^{3\eta}-\sum_{t=1}^{n}\delta_t+1\right)^{\frac{1}{\eta}}}\right),$$

$$FFSSPA(\hat{z_1},\hat{z_2},\ldots,\hat{z_n})$$
$$=\left(\sqrt[3]{1-\left(\sum_{t=1}^{n}\hat{\delta}_t(1-\hat{\alpha_t}^3)^{\eta}-\sum_{t=1}^{n}\hat{\delta}_t+1\right)^{\frac{1}{\eta}}},\right.$$
$$\left.\sqrt[3]{\left(\sum_{t=1}^{n}\hat{\delta}_t\hat{\beta_t}^{3\eta}-\sum_{t=1}^{n}\hat{\delta}_t+1\right)^{\frac{1}{\eta}}}\right).$$

Since $z_t\le\hat{z_t}$, $\forall t\ (t=1,2,\ldots,n)$ therefore we have $\alpha_t\le\hat{\alpha_t}$ and $\beta_t\ge\hat{\beta_t}\quad\forall\quad t\in 1,2,\ldots,n$, which implies that $\alpha_t^3\le\hat{\alpha}_t^3\Rightarrow(1-\alpha_t^3)\ge(1-\hat{\alpha}_t^3)$.

Because $\eta < 0$ therefore $(1-\alpha_t^3)^\eta \leq (1-\hat{\alpha}_t^3)^\eta$. Now, we have

$$\Rightarrow \left(\sum_{t=1}^{n} \delta_t (1-\alpha_t^3)^\eta - \sum_{t=1}^{n} \delta_t + 1\right)^{\frac{1}{\eta}}$$

$$\geq \left(\sum_{t=1}^{n} \hat{\delta}_t (1-\hat{\alpha}_t{}^3)^\eta - \sum_{t=1}^{n} \hat{\delta}_t + 1\right)^{\frac{1}{\eta}}$$

$$\Rightarrow 1 - \left(\sum_{t=1}^{n} \delta_t (1-\alpha_t^3)^\eta - \sum_{t=1}^{n} \delta_t + 1\right)^{\frac{1}{\eta}}$$

$$\leq 1 - \left(\sum_{t=1}^{n} \hat{\delta}_t (1-\hat{\alpha}_t{}^3)^\eta - \sum_{t=1}^{n} \hat{\delta}_t + 1\right)^{\frac{1}{\eta}}$$

$$\Rightarrow \sqrt[3]{1 - \left(\sum_{t=1}^{n} \delta_t (1-\alpha_t^3)^\eta - \sum_{t=1}^{n} \delta_t + 1\right)^{\frac{1}{\eta}}}$$

$$\leq \sqrt[3]{1 - \left(\sum_{t=1}^{n} \hat{\delta}_t (1-\hat{\alpha}_t{}^3)^\eta - \sum_{t=1}^{n} \hat{\delta}_t + 1\right)^{\frac{1}{\eta}}}.$$

Similarly, if $\hat{\beta}_t \leq \beta_t$, we obtain

$$\left(\sum_{t=1}^{n} \hat{\delta}_t \hat{\beta}_t{}^{3\eta} - \sum_{t=1}^{n} \hat{\delta}_t + 1\right)^{\frac{1}{\eta}} \leq \left(\sum_{t=1}^{n} \delta_t \beta_t^{3\eta} - \sum_{t=1}^{n} \delta_t + 1\right)^{\frac{1}{\eta}}$$

$$\Rightarrow \sqrt[3]{\left(\sum_{t=1}^{n} \hat{\delta}_t \hat{\beta}_t{}^{3\eta} - \sum_{t=1}^{n} \hat{\delta}_t + 1\right)^{\frac{1}{\eta}}} \leq \sqrt[3]{\left(\sum_{t=1}^{n} \delta_t \beta_t^{3\eta} - \sum_{t=1}^{n} \delta_t + 1\right)^{\frac{1}{\eta}}}.$$

Thus, we get $FFSSPA(z_1, z_2, \ldots, z_n) \leq FFSSPA(\hat{z_1}, \hat{z_2}, \ldots, \hat{z_n})$.

Property 3.3 (Boundedness). Let $z_1 = (\alpha_1, \beta_1)$, $z_2 = (\alpha_2, \beta_2), \ldots, z_n = (\alpha_n, \beta_n)$ be FFNs, $z^+ = (\max_{t=1}^n \alpha_t, \min_{t=1}^n \beta_t)$, $z^- = (\min_{t=1}^n \alpha_t, \max_{t=1}^n \beta_t)$. Then

$$z^- \leq \text{FFSSPA}(z_1, z_2, \ldots, z_n) \leq z^+.$$

Proof 3.4. Since $\min_{t=1}^{n} \alpha_t \leq \alpha_t \leq \max_{t=1}^{n} \alpha_t \quad \min_{t=1}^{n} \beta_t \leq \beta_t \leq \max_{t=1}^{n} \beta_t$, $\forall t \in 1, 2, \ldots, n$, then $z^- \leq z_t$ for $t = 1, 2, \ldots, n$. Thus, by using monotonicity property, we have

$$\text{FFSSPA}(z^-, z^-, \ldots, z^-) \leq \text{FFSSPA}(z_1, z_2, \ldots, z_n).$$

Now, by using idempotency property, we have

$$z^- \leq \text{FFSSPA}(z_1, z_2, \ldots, z_n). \tag{7}$$

Similarly,

$$\text{FFSSPA}(z_1, z_2, \ldots, z_n) \leq z^+. \tag{8}$$

Combining Equations (7) and (8), we get

$$z^- \leq \text{FFSSPA}(z_1, z_2, \ldots, z_n) \leq z^+.$$

3.1 The FFSSPWA Aggregation Operator

Definition 3.2. Let $z_1, z_2, \ldots, z_n$ be a collection of FFNs and $w_1, w_2, \ldots, w_n$ be the weights of $z_1, z_2, \ldots, z_n$, respectively, such that $w_t \geq 0$ and $\sum_{t=1}^{n} w_t = 1$. The proposed FFSSPWA AO for aggregating $z_1, z_2, \ldots, z_n$ is defined as follows:

$$FFSSPWA(z_1, z_2, \ldots, z_n) = \bigoplus_{t=1}^{n} \frac{w_t(1 + T(z_t))}{\sum_{t=1}^{n} w_t(1 + T(z_t))} z_t, \tag{9}$$

where $T(z_t) = \sum_{\substack{t=1 \\ t \neq k}}^{n} S(z_t, z_k)$, $S(z_t, z_k) = 1 - \left(\frac{1}{2} \left\{ \left(\sqrt{\alpha_t^3} - \sqrt{\alpha_k^3} \right)^2 + \left(\sqrt{\beta_t^3} - \sqrt{\beta_k^3} \right)^2 + \left(\sqrt{\pi_t^3} - \sqrt{\pi_k^3} \right)^2 \right\} \right)^{1/2}$ denotes the support between the FFNs z_t from z_k, which satisfying the following properties:

(i) $S(z_t, z_k) \in [0, 1]$,
(ii) $S(z_t, z_k) = S(z_k, z_t)$,
(iii) $S(z_t, z_k) \geq S(z_p, z_q)$ if $|z_t - z_k| < |z_p - z_q|$.

Theorem 3.2. *For the FFNs $z_1 = (\alpha_1, \beta_1)$, $z_2 = (\alpha_2, \beta_2)$, …, $z_n = (\alpha_n, \beta_n)$, their aggregated value by using the proposed FFSSPWA AO is a FFN and provided as follows:*

$$FFSSPWA(z_1, z_2, \ldots, z_n)$$

$$= \left(\sqrt[3]{1 - \left(\sum_{t=1}^{n} \rho_t \big(1 - \alpha_t^3\big)^{\eta} - \sum_{t=1}^{n} \rho_t + 1 \right)^{\frac{1}{\eta}}}, \right.$$

$$\left. \sqrt[3]{\left(\sum_{t=1}^{n} \rho_t \beta_t^{3\eta} - \sum_{t=1}^{n} \rho_t + 1 \right)^{\frac{1}{\eta}}} \right), \tag{10}$$

where $\eta < 0$, $\rho_t = \frac{w_t(1+T(z_t))}{\sum_{t=1}^{n} w_t(1+T(z_t))}$, w_t *is the weight of* z_t *such that* $w_t \geq 0$, $t = 1, 2, \ldots, n$, $\sum_{t=1}^{n} w_t = 1$ *and* $T(z_t) = \sum_{\substack{t=1 \\ t \neq k}}^{n} S(z_t, z_k)$.

Proof 3.5. The proof of this theorem is similar to the proof of Theorem 3.1.

Example 3.2. Let $z_1 = (0.6, 0.8)$, $z_2 = (0.9, 0.2)$ and $z_3 = (0.4, 0.5)$ be any three FFNs with weights $w_1 = 0.3, w_2 = 0.4$ and $w_3 = 0.3$, respectively. First, we calculate the support values $S(z_t, z_k)$ between the FFNs z_t and z_k as follows:

$$\begin{aligned} S(z_1, z_2) &= 1 - \left(\sqrt{\frac{1}{2} \begin{bmatrix} \left(\sqrt{(0.6)^3} - \sqrt{(0.9)^3} \right)^2 \\ + \left(\sqrt{(0.8)^3} - \sqrt{(0.2)^3} \right)^2 \\ + \left(\sqrt{(0.6479)^3} - \sqrt{(0.6407)^3} \right)^2 \end{bmatrix}} \right) \\ &= 0.4787, \end{aligned}$$

$$\begin{aligned} S(z_1, z_3) &= 1 - \left(\sqrt{\frac{1}{2} \begin{bmatrix} \left(\sqrt{(0.6)^3} - \sqrt{(0.4)^3} \right)^2 \\ + \left(\sqrt{(0.8)^3} - \sqrt{(0.5)^3} \right)^2 \\ + \left(\sqrt{(0.6479)^3} - \sqrt{(0.9326)^3} \right)^2 \end{bmatrix}} \right) \\ &= 0.6003, \end{aligned}$$

$$S(z_2, z_3) = 1 - \left(\sqrt{\frac{1}{2} \begin{bmatrix} \left(\sqrt{(0.9)^3} - \sqrt{(0.4)^3}\right)^2 \\ + \left(\sqrt{(0.2)^3} - \sqrt{(0.5)^3}\right)^2 \\ + \left(\sqrt{(0.6407)^3} - \sqrt{(0.9326)^3}\right)^2 \end{bmatrix}} \right)$$
$$= 0.4610.$$

After that, we calculate $T(z_1), T(z_2)$ and $T(z_3)$ of the FFNs z_1, z_2 and z_3, respectively, where $T(z_t) = \sum_{\substack{t=1 \\ t \neq k}}^{n} S(z_t, z_k)$,

$$T(z_1) = S(z_1, z_2) + S(z_1, z_3)$$
$$= 0.4787 + 0.6003 = 1.0790,$$
$$T(z_2) = S(z_2, z_1) + S(z_2, z_3)$$
$$= 0.4787 + 0.4610 = 0.9397,$$
$$T(z_3) = S(z_3, z_1) + S(z_3, z_2)$$
$$= 0.6003 + 0.4610 = 1.0613.$$

Now, we calculate ρ_1, ρ_2 and ρ_3 for the FFNs z_1, z_2 and z_3, respectively, where $\rho_1 = \frac{w_1(1+T(z_1))}{\sum_{t=1}^{3} w_t(1+T(z_t))} = 0.3091$, $\rho_2 = 0.3845$ and $\rho_3 = 0.3064$.

Now, by using Equation (10), we aggregate the FFNs z_1, z_2 and z_3, for $\eta = -3$, as follows:

$$\text{FFSSPWA}(z_1, z_2, z_3) = \left(\sqrt[3]{\begin{matrix} 1 - \Big(0.3091 \left(1 - (0.6)^3\right)^{-3} \\ + 0.3845 \left(1 - (0.9)^3\right)^{-3} \\ + 0.3064 \left(1 - (0.4)^3\right)^{-3} - 1 + 1 \Big)^{\frac{1}{-3}} \end{matrix}} \right.,$$

$$\sqrt[3]{\left(\begin{array}{l}\left(0.3091\,(0.8)^{3(-3)} + 0.3845\,(0.2)^{3(-3)}\right.\\ \left. + 0.3064\,(0.5)^{3(-3)} - 1 + 1\right)^{\frac{1}{-3}}\end{array}\right)},$$

$$= (0.8589, 0.2224).$$

Property 3.4 (Idempotency). Let $z_1 = (\alpha_1, \beta_1)$, $z_2 = (\alpha_2, \beta_2)$, ..., $z_n = (\alpha_n, \beta_n)$ be FFNs with weights $w_1, w_2, \ldots,$ and w_n, respectively, where $w_t \geq 0$ and $\sum_{t=1}^{n} w_t = 1$. If $z_1 = z_2 = \ldots = z_n = z = (\alpha, \beta)$, then FFSSPWA$(z_1, z_2, \ldots, z_n) = z$.

Proof 3.6.

$$\begin{aligned}
FFSSPWA(z_1, z_2, \ldots, z_n) &= \left(\sqrt[3]{1 - \left(\sum_{t=1}^{n} \rho_t \big(1 - \alpha_t^3)^{\eta} - \sum_{t=1}^{n} \rho_t + 1\right)^{\frac{1}{\eta}}},\right. \\
&\qquad \left.\sqrt[3]{\left(\sum_{t=1}^{n} \rho_t \beta_t^{3\eta} - \sum_{t=1}^{n} \rho_t + 1\right)^{\frac{1}{\eta}}}\right) \\
&= \left(\sqrt[3]{1 - \left(\sum_{t=1}^{n} \rho_t \big(1 - \alpha^3)^{\eta} - \sum_{t=1}^{n} \rho_t + 1\right)^{\frac{1}{\eta}}},\right. \\
&\qquad \left.\sqrt[3]{\left(\sum_{t=1}^{n} \rho_t \beta^{3\eta} - \sum_{t=1}^{n} \rho_t + 1\right)^{\frac{1}{\eta}}}\right) \\
&= \left(\sqrt[3]{1 - \big((1 - \alpha^3)^{\eta} - 1 + 1\big)^{\frac{1}{\eta}}},\right. \\
&\qquad \left.\sqrt[3]{(\beta^{3\eta} - 1 + 1)^{\frac{1}{\eta}}}\right) \\
&= (\alpha, \beta) = z.
\end{aligned}$$

Property 3.5 (Monotonicity). Let $z_t = (\alpha_t, \beta_t)$ and $\hat{z}_t = (\hat{\alpha}_t, \hat{\beta}_t)$, $t = 1, 2, \ldots, n$, be two collections of FFNs with weight w_t of FFN z_t, where

$w_t \geq 0$ and $\sum_{t=1}^{n} w_t = 1$. If $z_t \leq \hat{z}_t$, $\forall t$ $(t = 1, 2, \ldots, n)$ then

$$FFSSPWA(z_1, z_2, \ldots, z_n) \leq FFSSPWA(\hat{z_1}, \hat{z_2}, \ldots, \hat{z_n}).$$

Proof 3.7.

$$FFSSPWA(z_1, z_2, \ldots, z_n) = \left(\sqrt[3]{1 - \left(\sum_{t=1}^{n} \rho_t \big(1 - \alpha_t^3\big)^{\eta} - \sum_{t=1}^{n} \rho_t + 1 \right)^{\frac{1}{\eta}}}, \right.$$
$$\left. \sqrt[3]{\left(\sum_{t=1}^{n} \rho_t \beta_t^{3\eta} - \sum_{t=1}^{n} \rho_t + 1 \right)^{\frac{1}{\eta}}} \right)$$

$$FFSSPWA(\hat{z_1}, \hat{z_2}, \ldots, \hat{z_n}) = \left(\sqrt[3]{1 - \left(\sum_{t=1}^{n} \hat{\rho}_t \big(1 - \hat{\alpha}_t{}^3\big)^{\eta} - \sum_{t=1}^{n} \hat{\rho}_t + 1 \right)^{\frac{1}{\eta}}}, \right.$$
$$\left. \sqrt[3]{\left(\sum_{t=1}^{n} \hat{\rho}_t \hat{\beta}_t{}^{3\eta} - \sum_{t=1}^{n} \hat{\rho}_t + 1 \right)^{\frac{1}{\eta}}} \right)$$

Since $\alpha_t \leq \hat{\alpha}_t$, $\forall t \in 1, 2, \ldots, n$, $\alpha_t^3 \leq \hat{\alpha}_t^3 \Rightarrow (1 - \alpha_t^3) \geq (1 - \hat{\alpha}_t^3) \Rightarrow (1 - \alpha_t^3)^{\eta} \leq (1 - \hat{\alpha}_t^3)^{\eta}$, (since $\eta < 0$)

$$\Rightarrow \left(\sum_{t=1}^{n} \rho_t \big(1 - \alpha_t^3\big) \right) \leq \left(\sum_{t=1}^{n} \hat{\rho}_t \big(1 - \hat{\alpha}_t{}^3\big)^{\eta} \right)$$

$$\Rightarrow \left(\sum_{t=1}^{n} \rho_t \big(1 - \alpha_t^3\big)^{\eta} - \sum_{t=1}^{n} \rho_t + 1 \right)^{\frac{1}{\eta}}$$

$$\geq \left(\sum_{t=1}^{n} \hat{\rho}_t \big(1 - \hat{\alpha}_t{}^3\big)^{\eta} - \sum_{t=1}^{n} \hat{\rho}_t + 1 \right)^{\frac{1}{\eta}}$$

$$\Rightarrow 1 - \left(\sum_{t=1}^{n} \rho_t \big(1 - \alpha_t^3\big)^{\eta} - \sum_{t=1}^{n} \rho_t + 1 \right)^{\frac{1}{\eta}}$$

$$\leq\ 1-\left(\sum_{t=1}^{n}\hat{\rho}_t\big(1-\hat{\alpha_t}^3)^{\eta}-\sum_{t=1}^{n}\hat{\rho}_t+1\right)^{\frac{1}{\eta}}$$

$$\Rightarrow\ \sqrt[3]{1-\left(\sum_{t=1}^{n}\rho_t\big(1-\alpha_t^3)^{\eta}-\sum_{t=1}^{n}\rho_t+1\right)^{\frac{1}{\eta}}}$$

$$\leq\ \sqrt[3]{1-\left(\sum_{t=1}^{n}\hat{\rho}_t\big(1-\hat{\alpha_t}^3)^{\eta}-\sum_{t=1}^{n}\hat{\rho}_t+1\right)^{\frac{1}{\eta}}}$$

similarly, if $\hat{\beta}_t \leq \beta_t$, we obtain

$$\left(\sum_{t=1}^{n}\hat{\rho}_t\hat{\beta_t}^{3\eta}\right)\geq\left(\sum_{t=1}^{n}\rho_t\beta_t^{3\eta}\right)\quad(\text{since}\quad\eta<0)$$

$$\Rightarrow\left(\sum_{t=1}^{n}\hat{\rho}_t\hat{\beta_t}^{3\eta}-\sum_{t=1}^{n}\hat{\rho}_t+1\right)^{\frac{1}{\eta}}$$

$$\leq\left(\sum_{t=1}^{n}\rho_t\beta_t^{3\eta}-\sum_{t=1}^{n}\rho_t+1\right)^{\frac{1}{\eta}}$$

$$\Rightarrow\sqrt[3]{\left(\sum_{t=1}^{n}\hat{\rho}_t\hat{\beta_t}^{3\eta}-\sum_{t=1}^{n}\hat{\rho}_t+1\right)^{\frac{1}{\eta}}}$$

$$\leq\sqrt[3]{\left(\sum_{t=1}^{n}\rho_t\beta_t^{3\eta}-\sum_{t=1}^{n}\rho_t+1\right)^{\frac{1}{\eta}}}$$

Because $z_t \leq \hat{z}_t, \forall\ t = 1, 2, \ldots, n$, we get

$$FFSSPWA(z_1, z_2, \ldots, z_n) \leq FFSSPWA(\hat{z_1}, \hat{z_2}, \ldots, \hat{z_n}).$$

Property 3.6 (Boundedness). Let $z_1 = (\alpha_1, \beta_1)$, $z_2 = (\alpha_2, \beta_2)$, ..., $z_n = (\alpha_n, \beta_n)$ be FFNs with weights $w_1, w_2, \ldots,$ and w_n, respectively, where $w_t \geq 0$ and $\sum_{t=1}^{n} w_t = 1$, $z^+ = (\max_{t=1}^{n}\alpha_t, \min_{t=1}^{n}\beta_t)$, $z^- = (\min_{t=1}^{n}\alpha_t, \max_{t=1}^{n}\beta_t)$. Then

$$z^- \leq \text{FFSSPWA}(z_1, z_2, \ldots, z_n) \leq z^+.$$

Proof 3.8. Since $\min_{t=1}^{n} \alpha_t \leq \alpha_t \leq \max_{t=1}^{n} \alpha_t \quad \min_{t=1}^{n} \beta_t \leq \beta_t \leq \max_{t=1}^{n} \beta_t$, $\forall t \in 1, 2, \ldots, n$, then $z^- \leq z_t$ for $t = 1, 2, \ldots, n$. Thus, by using monotonicity property, we have

$$\text{FFSSPWA}(z_1^-, z_2^-, \ldots, z_n^-) \leq \text{FFSSPWA}(z_1, z_2, \ldots, z_n)$$

By using idempotency, we have

$$z^- \leq \text{FFSSPWA}(z_1, z_2, \ldots, z_n). \tag{11}$$

Similarly,

$$\text{FFSSPWA}(z_1, z_2, \ldots, z_n) \leq z^+. \tag{12}$$

Combining Equations (11) and (12), we get

$$z^- \leq \text{FFSSPWA}(z_1, z_2, \ldots, z_n) \leq z^+.$$

4 The Proposed MADM Algorithm Based on the Proposed FFSSPWA AO of FFNs

In this section, we propose a new MADM algorithm based on FFSSPWA AO under FFNs environment. Let $A_1, A_2, \ldots, A_m$ be m alternatives and $G_1, G_2, \ldots, G_n$ be n attributes with weights w_1, w_2, …, w_n, where $w_t \in [0, 1]$ and $\sum_{t=1}^{n} w_t = 1$. Expert assess the alternative A_k with respect to attribute G_t by utilizing the FFN $z_{kt} = (\alpha_{kt}, \beta_{kt})$, where $k = 1, 2, \ldots, m$ and $t = 1, 2, \ldots, n$, to construct the decision-matrix (DMx) $\tilde{D} = (\tilde{z}_{kt})_{m\times n}$=$(\langle\tilde{\alpha}_{kt}, \tilde{\beta}_{kt}\rangle)_{m\times n}$, shown as below:

$$\widetilde{D} = \begin{matrix} & G_1 & G_2 & \ldots & G_n \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \begin{pmatrix} \tilde{z}_{11} & \tilde{z}_{12} & \ldots & \tilde{z}_{1n} \\ \tilde{z}_{21} & \tilde{z}_{22} & \ldots & \tilde{z}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{z}_{m1} & \tilde{z}_{m2} & \ldots & \tilde{z}_{mn} \end{pmatrix},$$

The proposed MADM algorithm has the following steps:

Step 1: Convert the DMx $\widetilde{D} = (\tilde{z}_{kt})_{m\times n} = (\tilde{\alpha}_{kt}, \tilde{\beta}_{kt})_{m\times n}$ into normalized DMx (NDMx) $D = (z_{kt})_{m\times n} = (\alpha_{kt}, \beta_{kt})_{m\times n}$ as follows:

$$z_{kt} = \begin{cases} (\tilde{\alpha}_{kt}, \tilde{\beta}_{kt}) : \text{for benefit-type attribute} \\ (\tilde{\beta}_{kt}, \tilde{\alpha}_{kt}) : \text{ for cost-type attribute} \end{cases} \tag{13}$$

where $k = 1, 2, \ldots, m$ and $t = 1, 2, \ldots, n$.

Step 2: Calculate the support measure $S(z_{kt}, z_{kl})$ between the FFNs kt and kl as follows:

$$S(z_{kt}, z_{kl}) = 1 - \sqrt{\frac{1}{2}\left[\left(\sqrt{\alpha_{kt}^3} - \sqrt{\alpha_{kl}^3}\right)^2 + \left(\sqrt{\beta_{kt}^3} - \sqrt{\beta_{kl}^3}\right)^2 + \left(\sqrt{\pi_{kt}^3} - \sqrt{\pi_{kl}^3}\right)^2\right]}, \quad (14)$$

where $t, l = 1, 2, \ldots, n$ and $t \neq l$.

Step 3: Calculate the support value $T(z_{kt})$ corresponding to each FFN z_{kt} as follows:

$$T(z_{kt}) = \sum_{\substack{l,t=1 \\ l \neq t}}^{n} S(z_{kt}, z_{kl}). \quad (15)$$

Step 4: Based on the support value $T(z_{kt})$, we obtain the weight ρ_{kt} of the FFN z_{kt} as follows:

$$\rho_{kt} = \frac{w_t(1 + T(z_{kt}))}{\sum_{t=1}^{n} w_t(1 + T(z_{kt}))}. \quad (16)$$

Step 5: By utilizing the proposed FFSSPWA AO, we aggregate the FFNs z_{k1},z_{k2},. . ., z_{kn} to get the overall FFN $z_k = (\alpha_k, \beta_k)$ of the alternative A_k, shown as follows:

$$\begin{aligned} z_k &= FFSSPWA(z_{k1}, z_{k2}, \ldots, z_{kn}) \\ &= \left\langle \sqrt[3]{1 - \left(\sum_{t=1}^{n} \rho_t (1 - \alpha_t^3)^{\eta} - \sum_{t=1}^{n} \rho_t + 1\right)^{\frac{1}{\eta}}}, \sqrt[3]{\left(\sum_{t=1}^{n} \rho_t \beta_t^{3\eta} - \sum_{t=1}^{n} \rho_t + 1\right)^{\frac{1}{\eta}}} \right\rangle, \end{aligned} \quad (17)$$

where $\eta < 0$.

Step 6: By using Equations (2) and (3), we obtain the score value $s(z_k)$ and accuracy value $\Psi(z_k)$, respectively, of the overall FFN $z_k = (\alpha_k, \beta_k)$ of alternative A_k, shown as follows:

$$s(z_k) = \alpha_k^3 - \beta_k^3, \quad (18)$$

$$\Psi(z_k) = \alpha_k^3 + \beta_k^3. \quad (19)$$

Step 7: By using Definition 2.4, score values $s(z_1)$, $s(z_2)$, ..., $s(z_m)$ and accuracy values $\Psi(z_1)$, $\Psi(z_2)$, ..., $\Psi(z_m)$, obtain the ranking order (RO) of the alternatives A_1, A_2, ..., A_m, and choose the best alternative.

Figure 2 represents comprehensive flow chart of the proposed MADM algorithm.

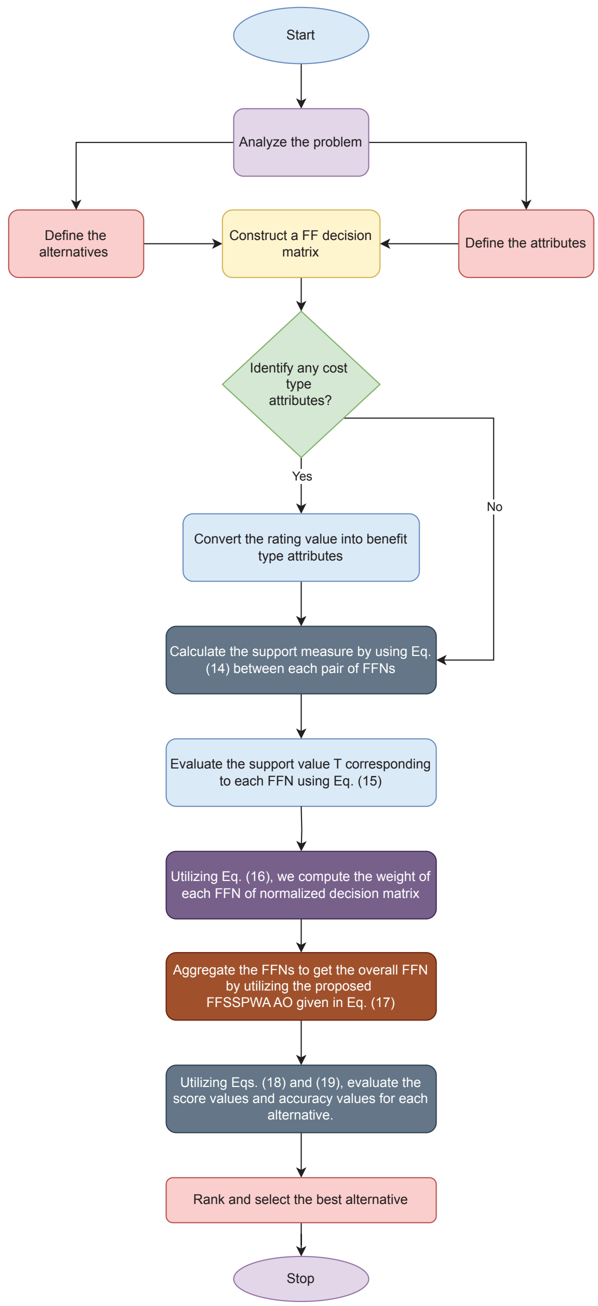

Figure 2 Flowchart of the proposed MADM algorithm.

5 Case Study of the Proposed MADM Algorithm

In this section, we consider a mathematical case study of selection of optimal healthcare waste management methods. This mathematical case study shows how much easier it is to apply the proposed MADM algorithm in real life to select the most appropriate HCWMT, which is a very important process to maintain public health and environmental security. Effective waste management has been identified as one of the most important practices in the healthcare industry over the past few years.

Example 5.1. Activities related to healthcare are essential for preserving and regaining health as well as saving lives. The trash and byproducts produced by these operations, however, present a serious problem. Like household garbage, around 85% of the waste generated by healthcare operations is ordinary, non-hazardous waste. The remaining 15% is classified as hazardous material, which might be radioactive, poisonous, infectious, combustible, corrosive, reactive, or carcinogenic [8]. Although approximately 16 billion vaccinations are given annually worldwide, not all syringes and needles are disposed away appropriately after use. It is crucial to take the right precautions for the safe and ecologically conscious handling of healthcare waste (HCW). These precautions aid in preventing possible environmental damage and health dangers, such as the unintentional discharge of biological or chemical threats. HCW and by-products consist of a diverse range of materials, such as:

Types of the Health Care Waste

In the following, this paper outlines the various types of health care waste [11, 26].

Infectious waste is the waste that is believed to contain pathogens and may spread disease. This refers to waste and wastewater containing blood and bodily fluids, along with extremely infectious materials like laboratory cultures, microbiological samples, and waste from items used with patients with highly contagious diseases in isolation units.

Medical waste: Includes human tissues, organs, body parts, fetuses, unused blood products, and contaminated animal carcasses.

Waste involving sharps: Includes both used and unused sharp items like hypodermic needles, IV needles, auto-disable syringes, syringes with needles, infusion sets, scalpels, pipettes, knives, blades, and shattered glass.

Figure 3 Graphical representation of the health care waste.

Chemical waste comprises of solvents and reagents utilized in lab experiments, disinfectants, sterilants, as well as heavy metals present in medical equipment (such as mercury in damaged thermometers) and batteries.

Pharmaceutical and cytotoxic waste includes pharmaceuticals that have expired or are not used, along with items that have been contaminated by or contain pharmaceuticals. Cytotoxic waste consists of substances that have genotoxic properties, like waste containing cytostatic drugs (often used for cancer treatment) or genotoxic chemicals.

Radioactive waste refers to items that have been tainted with radionuclides, such as radioactive diagnostic tools or radiotherapeutic drugs.

Non-dangerous or ordinary waste: Includes waste that lacks noteworthy biological, chemical, radioactive, or physical hazards.

Figure 3 represents the various types of the health care wastes.

Origins of the Health Care Wastes

In the following, some main sources of the generation of the health care waste are given:

1. Medical centers and other healthcare institutions

2. Labs and research facilities
3. Funeral homes and morgues
4. Laboratories conducting research and testing on animals
5. Services for collecting and storing blood
6. Care facilities for senior citizens

Health Care Waste Management Technologies

In developed nations, there is approximately 0.5 kg of hazardous waste produced daily per hospital bed, whereas in developing nations, the average is 0.2 kg per bed. Yet, in developing nations, health care waste is frequently not categorized into hazardous or non-hazardous wastes, leading to an inflated amount of hazardous waste. HCW has a detrimental impact on the environment through pollution, soil and water contamination, as well as the release of harmful emissions. Hence, it is crucial to have efficient waste management in order to minimize these negative effects. Proper waste segregation at the source, compliance with regulations, and comprehensive training for healthcare workers are vital in achieving this goal. Highlighting the importance of sustainable practices, such as recycling and reducing single-use items, is essential as well. Nevertheless, it is crucial to understand that recycling alone is not a complete answer to the problem. Investing in modern treatment technologies such as autoclaving and incineration is essential for efficient management of healthcare waste and reducing related environmental and health hazards. Following an initial assessment, five potential alternatives of healthcare waste management technologies (HCWMTs) are determined: " Mechanical Biological Treatment" (A_1), "Hydrothermal Carbonization" (A_2), "Incineration" (A_3), "Microwaving" (A_4), and "Chemical Disinfection" (A_5). A brief discussion about these HCWMTs is given below:

Alternatives Description

Mechanical Biological Treatment (MBT) (A_1) [25]: MBT is a waste processing technology combining mechanical sorting with biological treatment processes to reduce the volume of waste and recover useful materials, while minimizing the environmental impact of waste disposal. MBT in the management of municipalities and health care wastes will be effectively managed by diverting waste away from landfills as well as by recovering valuable resources such as energy, compost, or other recyclable material.

Hydrothermal Carbonization (HTC) (A_2) [22]: HTC is a promising technology that can be used to address many of the challenges associated with the management of HWM in healthcare facilities. Healthcare facilities produce a great deal of waste – while much of it is organic, such as food waste, paper, and some medical materials, it also includes hazardous materials, such as sharps, pharmaceutical wastes, and biohazardous materials. The HTC can be very useful for the treatment of specific types of healthcare waste, especially organic waste, and sewage sludge, where it reduces environmental impacts while promoting sustainability.

Incineration (A_3) [22]: Incineration is the process of burning the biomedical waste to convert it into gases and ash that are hard to burn again. The non-burnt ash left after the treatment is disposed of in a landfill. This combustion process can effectively break down waste materials, including elements like carbon, hydrogen, oxygen, sulfur, chlorine, and others, depending on the waste composition and the efficiency of the incinerator system. The process yields flue gases; these are generally harmless carbon dioxide, water, and nitrogen. However, it also emits poisonous emissions in the form of acidic gases such as sulfur oxides, acids, and nitrogen; toxic materials include heavy metals, dioxins, and furans. Incinerators incorporate air pollution control devices (APCDs) to ensure the best possible removal of contaminants from flue gases in order to minimize the impact of these pollutants on the environment. Waste volume is reduced by about 80-85% via incineration.

Microwaving (A_4) [27]: Microwave energy is deployed to treat medical wastes and wastewater sludge. To achieve a better microwave treatment, as well as to bring down the quantity of solid waste for disposal, the wastes are normally shredded and pre-moistened. A microwave unit directly heats the wet waste by deploying microwaves, unlike an autoclave that can be used to heat waste externally, just like a conventional oven. However, microwave treatment is not appropriate for contaminated animal carcasses, large metal objects, or hazardous waste such as cytotoxic, toxic, or radioactive materials. The advantage of microwaving lies in the fact that there is no requirement for steam and electricity usage is very low.

Chemical Disinfection (A_5) [16]: Chemical disinfection mainly applies to liquid waste including blood, urine, feces, or sewage from healthcare facilities. It can further be used to disinfect microbiological cultures, contaminated sharps, or shredded. Normally, chemical disinfectants such as sodium hypochlorite, calcium hypochlorite, and chlorine dioxide are mixed with the waste and left to act for a predetermined period. This method functions by breaking down organic matter and inactivating infectious microorganisms. Chemical disinfection is generally considered adequate when the volume of waste is relatively small.

Attributes Descriptions

HCWMTs for HCW management are evaluated using seven attributes "Environmental hazard" (G_1), "Health risk" (G_2), "Investment cost" (G_3), "Operation and maintenance cost" (G_4), "Revenue generation" (G_5), "Public acceptance" (G_6) and "Requirement of skilled labor" (G_7). These attributes are identified through literature review and discussion with experts. A brief summary of these attributes is given as follows:

(i) **"Environmental hazard"** (G_1) [16]: Environmental hazards serve as essential criteria for evaluating potential risks, including air pollutants, emissions of greenhouse gases, and toxic residues from burning or chemical processes. This assessment ensures the selection of HCWTTs that minimize harm to ecosystems and human health while optimizing energy efficiency.

(ii) **"Health risk"** (G_2) [25]: Potentially dangerous microbes found in HCW have the potential to infect HCW management personnel, and members of the public. Drug-resistant bacteria that escape from medical institutions and into the environment could be additional risks.

(iii) **"Investment cost"** (G_3) [11]: The cost of investment directly affects project feasibility, shaping budget allocation, financing alternatives, and return on investment assessments. This approach enables stake holders to prioritize cost-effective HCWTTs that enhance long-term financial benefits and promote sustainable waste management practices.

(iv) **"Operation and maintenance cost"** (G_4) [12]: Because they affect ongoing financial obligations, operational efficiency, and profitability during the facility's lifetime, operation and maintenance expenses are important economic factors.

(v) **"Revenue generation"** (G_5) [25]: HCW management may become a money-generating endeavor since revenue production balances initial investment and operating expenses. By utilizing resource recovery and energy production to improve economic viability and promote long-term financial sustainability in HCW management systems, it encourages sustainable behaviors.

(vi) **"Public acceptance"** (G_6) [26]: Considers the general public's approval and support of HCWT methods.

(vii) **"Requirement of skilled labor"** (G_7): [12] The need for skilled labor affects community empowerment, local economic growth, and job creation by providing chances for specialized training and employment. It fosters community-based sustainable livelihoods and improves workforce skills.

Furthermore, the HCWMTs A_1, A_2, A_3, A_4 and A_5 are evaluated under the seven attributes "Environmental hazard" (G_1), "Health risk" (G_2), "Investment cost" (G_3), "Operation and maintenance cost" (G_4), "Revenue generation" (G_5), "Public acceptance" (G_6) and "Requirement of skilled labor" (G_7) by the decision making expert (DMEx), where the weights of the attributes are $w_1 = 0.25, w_2 = 0.1, w_3 = 0.1, w_4 = 0.2, w_5 = 0.1, w_6 = 0.1$ and $w_7 = 0.15$, respectively. The DMEx uses a FFN $z_{kt} = (\alpha_{kt}, \beta_{kt})$ to assess the alternative A_k under the attribute G_t, to assemble the DMx $\tilde{D} = (\tilde{z}_{kt})_{m\times n}$=$(\langle\tilde{\alpha}_{kt}, \tilde{\beta}_{kt}\rangle)_{m\times n}$, shown as follows:

$$\tilde{D} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} G_1 & G_2 & G_3 & G_4 & G_5 & G_6 & G_7 \end{array} \\ \begin{pmatrix} \langle 0.2,0.2\rangle & \langle 0.3,0.3\rangle & \langle 0.1,0.6\rangle & \langle 0.5,0.4\rangle & \langle 0.5,0.5\rangle & \langle 0.5,0.6\rangle & \langle 0.2,0.7\rangle \\ \langle 0.1,0.4\rangle & \langle 0.3,0.5\rangle & \langle 0.2,0.6\rangle & \langle 0.4,0.3\rangle & \langle 0.3,0.5\rangle & \langle 0.1,0.1\rangle & \langle 0.4,0.7\rangle \\ \langle 0.5,0.3\rangle & \langle 0.4,0.5\rangle & \langle 0.2,0.7\rangle & \langle 0.4,0.6\rangle & \langle 0.4,0.6\rangle & \langle 0.3,0.3\rangle & \langle 0.9,0.2\rangle \\ \langle 0.8,0.1\rangle & \langle 0.7,0.3\rangle & \langle 0.4,0.5\rangle & \langle 0.3,0.4\rangle & \langle 0.2,0.8\rangle & \langle 0.6,0.1\rangle & \langle 0.3,0.1\rangle \\ \langle 0.1,0.8\rangle & \langle 0.1,0.9\rangle & \langle 0.2,0.3\rangle & \langle 0.5,0.4\rangle & \langle 0.1,0.4\rangle & \langle 0.3,0.3\rangle & \langle 0.6,0.2\rangle \end{pmatrix} \end{array}.$$

To solve this case study, we utilize the proposed MADM algorithm, which is illustrated below:

Step 1: Since the attributes G_1, G_2, G_3 and G_4 are cost kind, and G_5, G_6 and G_7 are benefit kind, therefore by using Equation (13), we obtain NDMx $D = (\tilde{z}_{kt})_{5\times 7} = (z_{kt})_{5\times 7}$, where

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} G_1 & G_2 & G_3 & G_4 & G_5 & G_6 & G_7 \end{array} \\ \begin{pmatrix} \langle 0.2,0.2\rangle & \langle 0.3,0.3\rangle & \langle 0.6,0.1\rangle & \langle 0.4,0.5\rangle & \langle 0.5,0.5\rangle & \langle 0.5,0.6\rangle & \langle 0.2,0.7\rangle \\ \langle 0.4,0.1\rangle & \langle 0.5,0.3\rangle & \langle 0.6,0.2\rangle & \langle 0.3,0.4\rangle & \langle 0.3,0.5\rangle & \langle 0.1,0.1\rangle & \langle 0.4,0.7\rangle \\ \langle 0.3,0.5\rangle & \langle 0.5,0.4\rangle & \langle 0.7,0.2\rangle & \langle 0.6,0.4\rangle & \langle 0.4,0.6\rangle & \langle 0.3,0.3\rangle & \langle 0.9,0.2\rangle \\ \langle 0.1,0.8\rangle & \langle 0.3,0.7\rangle & \langle 0.5,0.4\rangle & \langle 0.4,0.3\rangle & \langle 0.2,0.8\rangle & \langle 0.6,0.1\rangle & \langle 0.3,0.1\rangle \\ \langle 0.8,0.1\rangle & \langle 0.9,0.1\rangle & \langle 0.3,0.2\rangle & \langle 0.4,0.5\rangle & \langle 0.1,0.4\rangle & \langle 0.3,0.3\rangle & \langle 0.6,0.2\rangle \end{pmatrix} \end{array}.$$

Step 2: By using Equation (14), we calculate the support measures $S(z_{kt}, z_{kl})$ between the FFNs z_{kt} and z_{kl} as follows: $S(z_{11}, z_{12}) = 0.9239$,

$S(z_{11}, z_{13}) = 0.7210$,
$S(z_{11}, z_{14}) = 0.7710, S(z_{11}, z_{15}) = 0.7213$, $S(z_{11}, z_{16}) = 0.6514$, $S(z_{11}, z_{17}) = 0.6252$,
$S(z_{12}, z_{11}) = 0.9239, S(z_{12}, z_{13}) = 0.7596, S(z_{12}, z_{14}) = 0.8437, S(z_{12}, z_{15}) = 0.7963$,
$S(z_{12}, z_{16}) = 0.7244, S(z_{12}, z_{17}) = 0.6752, S(z_{13}, z_{11}) = 0.7210, S(z_{13}, z_{12}) = 0.7596$,
$S(z_{13}, z_{14}) = 0.7273, S(z_{13}, z_{15}) = 0.7588, S(z_{13}, z_{16}) = 0.6796, S(z_{13}, z_{17}) = 0.5235$,
$S(z_{14}, z_{11}) = 0.7710, S(z_{14}, z_{12}) = 0.8437, S(z_{14}, z_{13}) = 0.7273, S(z_{14}, z_{15}) = 0.9248$,
$S(z_{14}, z_{16}) = 0.8768, S(z_{14}, z_{17}) = 0.7883, S(z_{15}, z_{11}) = 0.7213, S(z_{15}, z_{12}) = 0.7963$,
$S(z_{15}, z_{13}) = 0.7588, S(z_{15}, z_{14}) = 0.9248, S(z_{15}, z_{16}) = 0.9125, S(z_{15}, z_{17}) = 0.7477$,
$S(z_{16}, z_{11}) = 0.6514, S(z_{16}, z_{12}) = 0.7244, S(z_{16}, z_{13}) = 0.6796, S(z_{16}, z_{14}) = 0.8768$,
$S(z_{16}, z_{15}) = 0.9125, S(z_{16}, z_{17}) = 0.7946, S(z_{17}, z_{11}) = 0.6252, S(z_{17}, z_{12}) = 0.6752$,
$S(z_{17}, z_{13}) = 0.5235, S(z_{17}, z_{14}) = 0.7883, S(z_{17}, z_{15}) = 0.7477, S(z_{17}, z_{16}) = 0.7986$,
$S(z_{21}, z_{22}) = 0.8778, S(z_{21}, z_{23}) = 0.8333, S(z_{21}, z_{24}) = 0.8311, S(z_{21}, z_{25}) = 0.7616$,
$S(z_{21}, z_{26}) = 0.8418, S(z_{21}, z_{27}) = 0.5842, S(z_{22}, z_{21}) = 0.8778, S(z_{22}, z_{23}) = 0.9011$,
$S(z_{22}, z_{24}) = 0.8504, S(z_{22}, z_{25}) = 0.8108, S(z_{22}, z_{26}) = 0.7477, S(z_{22}, z_{27}) = 0.6757$,
$S(z_{23}, z_{21}) = 0.8333, S(z_{23}, z_{22}) = 0.9011, S(z_{23}, z_{24}) = 0.7527, S(z_{23}, z_{25}) = 0.7157$,
$S(z_{23}, z_{26}) = 0.6799, S(z_{23}, z_{27}) = 0.6105, S(z_{24}, z_{21}) = 0.8311, S(z_{24}, z_{22}) = 0.8504$,
$S(z_{24}, z_{23}) = 0.7527, S(z_{24}, z_{25}) = 0.9253, S(z_{24}, z_{26}) = 0.8147, S(z_{24}, z_{27}) = 0.7242$,
$S(z_{25}, z_{21}) = 0.7616, S(z_{25}, z_{22}) = 0.8108, S(z_{25}, z_{23}) = 0.7157, S(z_{25}, z_{24}) = 0.9253$,
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$S(z_{31}, z_{34}) = 0.7702, S(z_{31}, z_{35}) = 0.8872, S(z_{31}, z_{36}) = 0.8613, S(z_{31}, z_{37}) = 0.4035,$
$S(z_{32}, z_{31}) = 0.8478, S(z_{32}, z_{33}) = 0.7883, S(z_{32}, z_{34}) = 0.9132, S(z_{32}, z_{35}) = 0.8302,$
$S(z_{32}, z_{36}) = 0.8437, S(z_{32}, z_{37}) = 0.5378, S(z_{33}, z_{31}) = 0.6390, S(z_{33}, z_{32}) = 0.7883,$
$S(z_{33}, z_{34}) = 0.8530, S(z_{33}, z_{35}) = 0.6441, S(z_{33}, z_{36}) = 0.6752, S(z_{33}, z_{37}) = 0.7193,$
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$S(z_{41}, z_{42}) = 0.8522, S(z_{41}, z_{43}) = 0.5765, S(z_{41}, z_{44}) = 0.5427, S(z_{41}, z_{45}) = 0.9590,$
$S(z_{41}, z_{46}) = 0.4125, S(z_{41}, z_{47}) = 0.4669, S(z_{42}, z_{41}) = 0.8522, S(z_{42}, z_{43}) = 0.7190,$
$S(z_{42}, z_{44}) = 0.6753, S(z_{42}, z_{45}) = 0.8722, S(z_{42}, z_{46}) = 0.5497, S(z_{42}, z_{47}) = 0.5853,$
$S(z_{43}, z_{41}) = 0.5765, S(z_{43}, z_{42}) = 0.7190, S(z_{43}, z_{44}) = 0.8981, S(z_{43}, z_{45}) = 0.5957,$
$S(z_{43}, z_{46}) = 0.8245, S(z_{43}, z_{47}) = 0.7854, S(z_{44}, z_{41}) = 0.5427, S(z_{44}, z_{42}) = 0.6753,$
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$S(z_{45}, z_{46}) = 0.4319, S(z_{45}, z_{47}) = 0.4712, S(z_{46}, z_{41}) =$

$0.4125, S(z_{46}, z_{42}) = 0.5497,$
$S(z_{46}, z_{43}) = 0.8245, S(z_{46}, z_{44}) = 0.8168, S(z_{46}, z_{45}) = 0.4319, S(z_{46}, z_{47}) = 0.7759,$
$S(z_{47}, z_{41}) = 0.4669, S(z_{47}, z_{42}) = 0.5853, S(z_{47}, z_{43}) = 0.7854, S(z_{47}, z_{44}) = 0.8848,$
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$S(z_{52}, z_{56}) = 0.4091, S(z_{52}, z_{57}) = 0.6223, S(z_{53}, z_{51}) = 0.5595, S(z_{53}, z_{52}) = 0.4114,$
$S(z_{53}, z_{54}) = 0.7947, S(z_{53}, z_{55}) = 0.8507, S(z_{53}, z_{56}) = 0.9466, S(z_{53}, z_{57}) = 0.7758,$
$S(z_{54}, z_{51}) = 0.5765, S(z_{54}, z_{52}) = 0.4478, S(z_{54}, z_{53}) = 0.7947, S(z_{54}, z_{55}) = 0.8218,$
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$S(z_{55}, z_{53}) = 0.8507, S(z_{55}, z_{54}) = 0.8218, S(z_{55}, z_{56}) = 0.8871, S(z_{55}, z_{57}) = 0.6670,$
$S(z_{56}, z_{51}) = 0.5545, S(z_{56}, z_{52}) = 0.4091, S(z_{56}, z_{53}) = 0.9466, S(z_{56}, z_{54}) = 0.8437,$
$S(z_{56}, z_{55}) = 0.8871, S(z_{56}, z_{57}) = 0.7717, S(z_{57}, z_{51}) = 0.7767, S(z_{57}, z_{52}) = 0.6223,$
$S(z_{57}, z_{53}) = 0.7758, S(z_{57}, z_{54}) = 0.7602, S(z_{57}, z_{55}) = 0.6670, S(z_{57}, z_{56}) = 0.7717.$

Step 3: By using Equation (15), we obtain the support value of $T(z_{kt})$ of the FFNs z_{kt} as follows:

$$T(z_{11}) = 4.4138, T(z_{12}) = 4.7230, T(z_{13}) = 4.1698, T(z_{14}) = 4.9319,$$
$$T(z_{15}) = 4.8614, T(z_{16}) = 4.6392, T(z_{17}) = 4.1544, T(z_{21}) = 4.7299,$$
$$T(z_{22}) = 4.8635, T(z_{23}) = 4.4933, T(z_{24}) = 4.8984, T(z_{25}) = 4.7555,$$
$$T(z_{26}) = 4.3799, T(z_{27}) = 3.9372, T(z_{31}) = 4.4090, T(z_{32}) = 4.7609,$$
$$T(z_{33}) = 4.3189, T(z_{34}) = 4.7051, T(z_{35}) = 4.3571, T(z_{36}) = 4.3152,$$
$$T(z_{37}) = 3.1365, T(z_{41}) = 3.8099, T(z_{42}) = 4.2538, T(z_{43}) = 4.3992,$$
$$T(z_{44}) = 4.3713, T(z_{45}) = 3.8836, T(z_{46}) = 3.8112, T(z_{47}) = 3.9696,$$
$$T(z_{51}) = 3.7649, T(z_{52}) = 3.0511, T(z_{53}) = 4.3386, T(z_{54}) = 4.2447,$$

$T(z_{55}) = 4.0037, T(z_{56}) = 4.4127, T(z_{57}) = 4.3737.$

Step 4: By using Equation (16), we calculate the weight ρ_{kt} of the FFN z_{kt}, where

$$\rho_{11} = 0.2438, \rho_{12} = 0.1031, \rho_{13} = 0.0931, \rho_{14} = 0.2137,$$
$$\rho_{15} = 0.1056, \rho_{16} = 0.1016, \rho_{17} = 0.1393, \rho_{21} = 0.2557,$$
$$\rho_{22} = 0.1047, \rho_{23} = 0.0981, \rho_{24} = 0.2106, \rho_{25} = 0.1027,$$
$$\rho_{26} = 0.0960, \rho_{27} = 0.1322, \rho_{31} = 0.2557, \rho_{32} = 0.1089,$$
$$\rho_{33} = 0.1006, \rho_{34} = 0.2157, \rho_{35} = 0.1013, \rho_{36} = 0.1005,$$
$$\rho_{37} = 0.1173, \rho_{41} = 0.2378, \rho_{42} = 0.1039, \rho_{43} = 0.1068,$$
$$\rho_{44} = 0.2124, \rho_{45} = 0.0966, \rho_{46} = 0.0951, \rho_{47} = 0.1474,$$
$$\rho_{51} = 0.2370, \rho_{52} = 0.0806, \rho_{53} = 0.1062, \rho_{54} = 0.2087,$$
$$\rho_{55} = 0.0995, \rho_{56} = 0.1077, \rho_{57} = 0.1604.$$

Step 5: By utilizing proposed FFSSPWA AO given in Equation (17), we obtain the overall FFN z_k of the alternatives A_k, where $\eta = -2$, $z_1 = \langle 0.4182, 0.1475\rangle$, $z_2 = \langle 0.4198, 0.1189\rangle$, $z_3 = \langle 0.7393, 0.2550\rangle$, $z_4 = \langle 0.3975, 0.1266\rangle$, and $z_5 = \langle 0.7456, 0.1208\rangle$.

Step 6: By using Equation (18), we obtain the score values $s(z_1) = 0.0699$, $s(z_2) = 0.0723$, $s(z_3) = 0.3875$, $s(z_4) = 0.0608$, $s(z_5) = 0.4127$ of the alternatives A_1, A_2, A_3, A_4 and A_5, respectively.

Step 7: Because $s(z_5) > s(z_3) > s(z_2) > s(z_1) > s(z_4)$ where $s(z_1) = 0.0699, s(z_2) = 0.0723, s(z_3) = 0.3875, s(z_4) = 0.0608$ and $s(z_5) = 0.4127$, the RO of the alternatives A_1, A_2, A_3, A_4 and A_5 is $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$. Therefore, "Chemical Disinfection" (A_5) is the best HCWMT among the " Mechanical Biological Treatment" (A_1), "Hydrothermal Carbonization" (A_2), "Incineration" (A_3), "Microwaving" (A_4), and "Chemical Disinfection" (A_5) for this mathematical case study.

5.1 Comparative Analysis with Existing MADM Algorithms

In the following, we compare the ROs of the HCWMTs " Mechanical Biological Treatment" (A_1), "Hydrothermal Carbonization" (A_2), "Incineration" (A_3), "Microwaving" (A_4), and "Chemical Disinfection" (A_5) for the Example 5.1 obtained by the proposed MADM algorithm with the Senapati and Yager's [29] MADM algorithms based on FFWA AO and FFPWA

Table 1 A comparative analysis of ROs of the alternatives obtained by various MADM algorithms for Example 5.1

	MADM Algorithms	ROs
M1	MADM algorithm based on FFWA AO [29]	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
M2	MADM algorithm based on FFPWA AO [29]	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
M3	MADM algorithm based on FFOWA AO [2]	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
M4	MADM algorithm based on FFYWA AO [14]	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
M	Proposed MADM algorithm	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$

AO, Alghazzawi et al.'s [2] MADM algorithm based on FFOWA AO, Garg et al.'s [14] MADM algorithm based on FFYWA AO to show the practical applicability and validity of the proposed MADM algorithm.

Table 1 and Figure 4 present a comparison of the ROs of HCWMTs " Mechanical Biological Treatment" (A_1), "Hydrothermal Carbonization" (A_2), "Incineration" (A_3), "Microwaving" (A_4), and "Chemical Disinfection" (A_5) obtained by the proposed MADM algorithm, Senapati and Yager's [29] MADM algorithms based on FFWA AO and FFPWA AO, Alghazzawi et al.'s [2] MADM algorithm based on FFOWA AO, Garg et al.'s [14] MADM algorithm based on FFYWA AO for the Example 5.1.

It is clear from Table 1 and Figure 4 that the proposed MADM algorithm, Senapati and Yager's [29] MADM algorithms based on FFWA AO and FFPWA AO, Alghazzawi et al.'s [2] MADM algorithm based on FFOWA AO,

Figure 4 Graphical comparison of ROs obtained by various MADM algorithms for Example 5.1.

Garg et al.'s [14] MADM algorithm based on FFYWA AO obtain the same RO "$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$" of the HCWMTs " Mechanical Biological Treatment" (A_1), "Hydrothermal Carbonization" (A_2), "Incineration" (A_3), "Microwaving" (A_4), and "Chemical Disinfection" (A_5). Therefore, "Chemical Disinfection" (A_5) is the best HCWMT among the " Mechanical Biological Treatment" (A_1), "Hydrothermal Carbonization" (A_2), "Incineration" (A_3), "Microwaving" (A_4), and "Chemical Disinfection" (A_5) for the sustainable HCW management in the case study given in Example 5.1. This validates the reliability and robustness of the proposed MADM algorithm.

6 Advantages of the Proposed MADM Algorithm Over the Existing MADM Algorithms

In the following, we consider two numerical MADM examples to highlights the superiority and efficacy of the proposed MADM algorithm as compared to existing MADM algorithms presented by Senapati and Yager [29], Alghazzawi et al. [2], and Garg et al. [14].

Example 6.1. Let A_1, A_2, A_3 and A_4 be four alternatives, and G_1, G_2, G_3 and G_4 be four benefit type attributes with the weights $w_1 = 0.4$, $w_2 = 0.3$, $w_3 = 0.2$ and $w_4 = 0.1$. DMEx evaluates the alternatives A_1, A_2, A_3 and A_4 with respect to attributes G_1, G_2, G_3 and G_4 by using FFNs z_{kt} to obtain the DMx $\widetilde{D} = (\tilde{z_{kt}})_{4\times 4} = (\tilde{\alpha_{kt}}, \tilde{\beta_{kt}})_{4\times 4}$ as shown follows:

$$\widetilde{D} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad G_1 \quad & \quad G_2 \quad & \quad G_3 \quad & \quad G_4 \quad \end{array} \\ \begin{pmatrix} \langle 0.2, 0.2\rangle & \langle 0.4, 0.6\rangle & \langle 0.3, 0.7\rangle & \langle 0.1, 0.9\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.5, 0.5\rangle & \langle 0.6, 0.4\rangle & \langle 0.8, 0.2\rangle \\ \langle 0.9, 0.1\rangle & \langle 0.4, 0.6\rangle & \langle 0.6, 0.4\rangle & \langle 0.7, 0.3\rangle \\ \langle 0.6, 0.4\rangle & \langle 0.8, 0.2\rangle & \langle 0.7, 0.3\rangle & \langle 0.5, 0.5\rangle \end{pmatrix} \end{array}$$

To solve this MADM problem, we use the proposed MADM algorithm, Senapati and Yager's [29] MADM algorithms based on FFWA AO and FFPWA AO, Alghazzawi et al.'s [2] MADM algorithm based on FFOWA AO, Garg et al.'s [14] MADM algorithm based on FFYWA AO, and obtained ROs are summarized in Table 2 and Figure 5.

It is clear from Table 2 and Figure 5 that the Senapati and Yager's [29] MADM algorithm based on FFWA AO obtains the RO "$A_3 = A_4 \succ A_2 \succ A_1$", where it cannot distinguish the RO between the alternatives A_3 and A_4. Similarly, Alghazzawi et al.'s [2] MADM algorithm based on FFOWA

Table 2 A comparative analysis of ROs of the alternatives obtained by various MADM algorithms for Example 6.1

	MADM Algorithms	ROs
M1	MADM algorithm based on FFWA AO [29]	$A_3 = A_4 \succ A_2 \succ A_1$
M2	MADM algorithm based on FFPWA AO [29]	$A_3 \succ A_4 \succ A_2 \succ A_1$
M3	MADM algorithm based on FFOWA AO [2]	$A_3 \succ A_4 = A_2 \succ A_1$
M4	MADM algorithm based on FFYWA AO [14]	$A_3 \succ A_4 \succ A_2 \succ A_1$
M	Proposed MADM algorithm	$A_3 \succ A_4 \succ A_2 \succ A_1$

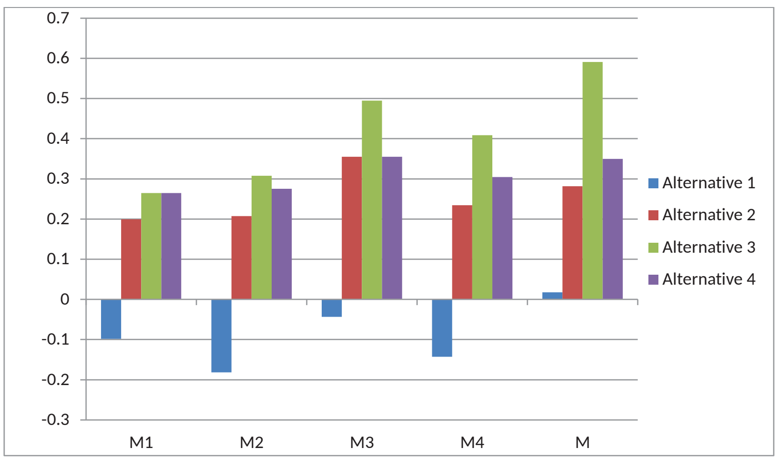

Figure 5 Graphical comparison of ROs derived by various MADM algorithms for Example 6.1.

AO obtains the RO "$A_3 \succ A_4 = A_2 \succ A_1$", where it cannot distinguish the RO between the alternatives A_2 and A_4. While, the proposed MADM algorithm, Garg et al.'s [14] MADM algorithm based on FFYWA AO and Senapati and Yager's [29] MADM algorithm based on FFPWA AO obtain the same RO "$A_3 \succ A_4 \succ A_2 \succ A_1$" for the alternatives A_1,A_2, A_3 and A_4. Therefore, the proposed MADM approach can overcome the limitations of Senapati and Yager's [29] MADM algorithm based on FFWA AO and Alghazzawi et al.'s [2] MADM algorithm based on FFOWA AO in this case.

Example 6.2. Let A_1, A_2, A_3 and A_4 be four alternatives, and G_1, G_2, G_3 and G_4 be four benefit type attributes with the weights $w_1 = 0.3$, $w_2 = 0.2$, $w_3 = 0.4$ and $w_4 = 0.1$. DMEx evaluates the alternatives A_1, A_2, A_3 and A_4 with respect to attributes G_1, G_2, G_3 and G_4 by using FFNs z_{kt} to obtain the

DMx $\widetilde{D} = (\tilde{z_{kt}})_{4\times4} = (\tilde{\alpha_{kt}}, \tilde{\beta_{kt}})_{4\times4}$ as shown follows:

$$\widetilde{D} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\ \begin{pmatrix} \langle 0.6, 0.7\rangle & \langle 0.763, 0.409\rangle & \langle 0.36, 0.6\rangle & \langle 0.62, 0.709\rangle \\ \langle 0.8, 0.4\rangle & \langle 0.3, 0.6\rangle & \langle 0.6, 0.3\rangle & \langle 0.7, 0.8\rangle \\ \langle 0.706, 0.52\rangle & \langle 0.5, 0.72\rangle & \langle 0.5, 0.497\rangle & \langle 0.8, 0.7\rangle \\ \langle 0.6, 0.697\rangle & \langle 0.381, 0.08\rangle & \langle 0.695, 0.491\rangle & \langle 0.6, 0.8\rangle \end{pmatrix} \end{array}$$

To solve this MADM problem, we use the proposed MADM algorithm, Senapati and Yager's [29] MADM algorithms based on FFWA AO and FFPWA AO, Alghazzawi et al.'s [2] MADM algorithm based on FFOWA AO, Garg et al.'s [14] MADM algorithm based on FFYWA AO, and obtained ROs are summarized in Table 3 and Figure 6.

Table 3 A comparative analysis of ROs of the alternatives obtained by various MADM algorithms for Example 6.2

	MADM Algorithms	ROs
M1	MADM algorithm based on FFWA AO [29]	$A_2 \succ A_4 \succ A_3 \succ A_1$
M2	MADM algorithm based on FFPWA AO [29]	$A_2 \succ A_4 = A_3 \succ A_1$
M3	MADM algorithm based on FFOWA AO [2]	$A_2 \succ A_4 \succ A_3 \succ A_1$
M4	MADM algorithm based on FFYWA AO [14]	$A_2 \succ A_4 = A_3 \succ A_1$
M	Proposed MADM algorithm	$A_2 \succ A_4 \succ A_3 \succ A_1$

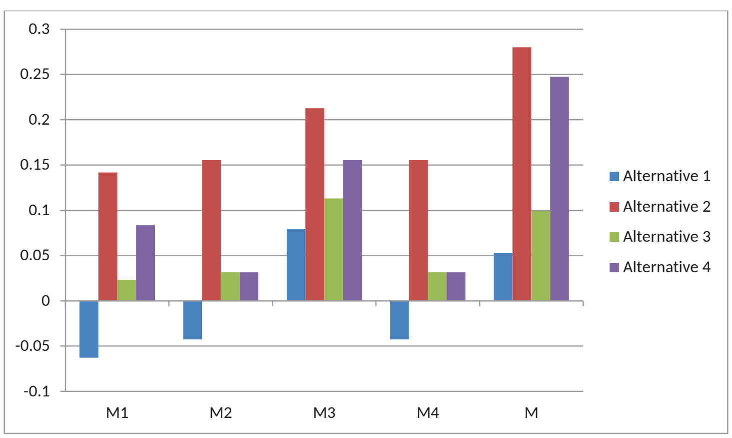

Figure 6 Graphical comparison of ROs derived by various MADM algorithms for Example 6.2.

It is clear from Table 3 and Figure 6 that the Senapati and Yager [29]'s MADM algorithm based on FFPWA AO and Garg et al.'s [14] MADM algorithm based on FFYWA AO obtain the same RO "$A_2 \succ A_4 = A_3 \succ A_1$" of the alternatives A_1, A_2, A_3 and A_4, where these MADM algorithms cannot distinguish the RO between the alternatives A_3 and A_4. While, the proposed MADM approach, Alghazzawi et al.'s [2] MADM algorithm based on FFOWA AO and Senapati and Yager's [29] MADM algorithm based on FFWA AO obtain the same RO "$A_2 \succ A_4 \succ A_3 \succ A_1$" for the alternatives A_1, A_2, A_3 and A_4. Therefore, the proposed MADM algorithm can overcome the limitations of Senapati and Yager's [29] MADM algorithm based on FFPWA AO and Garg et al.'s [14] MADM algorithm based on FFYWA AO in this case.

7 Conclusion

The health care waste (HCW) generated from the hospitals and other health care facilities create a high risks to health care personnel, patients, the public, and the environment. Therefore, a suitable sustainable management of the HCW is necessary to reduce the human and environmental risk. For this, the selection of the optimal HCW management technique (HCWMT) is an important task. Therefore, we have proposed a MADM algorithm in the context of FFNs to find the best sustainable HCWMT. For this, we have proposed FFSSPA AO and FFSSPWA AO for aggregating the FFNs based on the power averaging AO and Schweizer-Sklar's norm. Moreover, we have proposed a MADM algorithm based on the proposed FFSSPWA AO for FFNs. Afterwards, we have illustrated the proposed MADM algorithm by conducting a mathematical case study for the assessment of sustainable HCWMTs that considered the five HCWMTs: "Mechanical Biological Treatment", "Hydrothermal Carbonization", "Incineration", "Microwaving", and "Chemical Disinfection" to show the applicability of the proposed MADM algorithm. To show the robustness and validity of the results, we have compared the RO of the alternatives obtained using the proposed MADM algorithm with the ROs obtained from the existing MADM algorithms. From Example 6.1 and Example 6.2, it is clear that the proposed MADM algorithm can overcome the disadvantages of the MADM algorithm based on FFWA AO [29], MADM algorithm based on FFPWA AO [29], MADM algorithm based on FFOWA AO [2] and MADM algorithm based on FFYWA AO [14], which are unable to distinguish the ROs of the alternatives. The proposed FFSSPA AO, FFSSPWA AO, and proposed MADM algorithm fail to account for the priority order among various attributes, despite its objective existence in numerous real-world scenarios. Various generalizations and extensions of the FFNs also exist in the literature. Apart from that, the proposed MADM approach cannot solve the group decision making problems whereas group decision-making is frequently involved in real-life scenarios, the study solely considers a single decision-maker. Therefore, in the future, the proposed FFSSPA AO, FFSSPWA AO, and proposed MADM algorithm will be extended to other environments such as p, q-quasirung orthopair fuzzy sets [27], neutrosophic sets [13], bipolar complex fuzzy sets [21], probabilistic linguistic sets [15], etc., by integrating some innovative features such as prioritized, partitioned, and induced information. We can also expand this in the future to include a group decision-making process based on the proposed FFSSPWA AO in the FFNs environment.

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